# Solutions

8th Annual
Boston Area Undergraduate
Physics Competition

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1. (a) If $A$ is heated, then water will flow from $B$ to $A$. The reason can be seen as follows. The pressure at depth $h$ is given by $P = \rho g h$. When the water in $A$ expands, the height $h$ increases, but the density $\rho$ decreases. What happens to the product $\rho h$? The density goes like $1/A$, where $A$ is the area of the trapezoidal cross section. But $A = wh$, where $w$ is the width at half height. Therefore, $P = \rho g h \propto h/A = 1/w$. And since $w$ increases as the water level rises, the pressure in $A$ decreases, and water flows from $B$ to $A$.

   (b) If $B$ is heated, then water will again flow from $B$ to $A$. The same reasoning used above works here, except than now the $w$ in container $B$ *decreases*, so that the pressure in $B$ *increases*, so that the water again flows from $B$ to $A$.

2. Let $F$ be the tension in the string. The angle (at the mass) between the string and the radius of the dotted circle is $\theta = \sin^{-1}(r/R)$. In terms of $\theta$, the radial and tangential $F = ma$ equations are

$$\begin{aligned} F\cos\theta &= mv^2/R, \qquad \text{and} \\ F\sin\theta &= m\dot{v}. \end{aligned} \tag{1}$$

Solving for $F$ in the second equation and substituting into the first gives

$$\frac{m\dot{v}\cos\theta}{\sin\theta} = \frac{mv^2}{R}\,. \tag{2}$$

Separating variables and integrating gives

$$\begin{aligned} \int_{v_0}^{v} \frac{dv}{v^2} &= \frac{\tan\theta}{R}\int_0^t dt \\ \Longrightarrow \quad \frac{1}{v_0} - \frac{1}{v} &= \frac{\tan\theta}{R}t \\ \Longrightarrow \quad v &= \left(\frac{1}{v_0} - \frac{\tan\theta}{R}t\right)^{-1}. \end{aligned} \tag{3}$$

Note that $v$ becomes infinite when

$$t = T \equiv \frac{R}{v_0\tan\theta}\,. \tag{4}$$

In other words, you can keep the mass moving in the desired circle only up to time $T$. After that, it is impossible. (Of course, it will become impossible, for all practical purposes, long before $v$ becomes infinite.)

The total distance, $d = \int v\,dt$, is infinite, because this integral (barely) diverges (like a log), as $t$ approaches $T$.

3. Let $V$ be the initial speed. The horizontal speed and initial vertical speed are then $V\cos\theta$ and $V\sin\theta$, respectively. You can easily show that the distance traveled in the air is the standard

$$d_{\text{air}} = \frac{2V^2 \sin\theta\cos\theta}{g}\,. \tag{5}$$

To find the distance traveled along the ground, we must determine the horizontal speed just after the impact has occurred. The normal force, $N$, from the ground is what reduces the vertical speed from $V\sin\theta$ to zero, during the impact. So we have

$$\int N\,dt = mV\sin\theta, \tag{6}$$

where the integral runs over the time of the impact. But this normal force (when multiplied by $\mu$, to give the horizontal friction force) also produces a sudden decrease in the horizontal speed, during the time of the impact. So we have

$$m\Delta v_x = -\int(\mu N)\,dt = -\mu mV\sin\theta \qquad \Longrightarrow \qquad \Delta v_x = -\mu V\sin\theta. \tag{7}$$

(We have neglected the effect of the $mg$ gravitational force during the short time of the impact, since it is much smaller than the $N$ impulsive force.) Therefore, the brick begins its sliding motion with speed

$$v = V\cos\theta - \mu V\sin\theta. \tag{8}$$

Note that this is true only if $\tan\theta \le 1/\mu$. If $\theta$ is larger than this, then the horizontal speed simply becomes zero, and the brick moves no further. (Eq. (8) would give a negative value for $v$.)

The friction force from this point on is $\mu mg$, so the acceleration is $a = -\mu g$. The distance traveled along the ground can easily be shown to be

$$d_{\text{ground}} = \frac{(V\cos\theta - \mu V\sin\theta)^2}{2\mu g}\,. \tag{9}$$

We want to find the angle that maximizes the total distance, $d_{\text{total}} = d_{\text{air}} + d_{\text{ground}}$. From eqs. (5) and (9) we have

$$\begin{aligned} d_{\text{total}} &= \frac{V^2}{2\mu g}\left(4\mu\sin\theta\cos\theta + (\cos\theta - \mu\sin\theta)^2\right) \\ &= \frac{V^2}{2\mu g}(\cos\theta + \mu\sin\theta)^2. \end{aligned} \tag{10}$$

Taking the derivative with respect to $\theta$, we see that the maximum total distance is achieved when

$$\tan\theta = \mu. \tag{11}$$

Note, however, that the above analysis is valid only if $\tan\theta \le 1/\mu$ (from the comment after eq. (8)). We therefore see that if:

- $\mu < 1$, then the optimal angle is given by $\tan\theta = \mu$. (The brick continues to slide after the impact.)

- $\mu \geq 1$, then the optimal angle is $\theta = 45°$. (The brick stops after the impact, and $\theta = 45°$ gives the maximum value for the $d_{\text{air}}$ expression in eq. (5).)

4. The key point in this problem is that the sheet expands about a certain stationary point, but contracts around another (so that it ends up moving down the roof like an inchworm). We must find the locations of these two points.

   Let's consider the expansion first. Let the stationary point be a distance $a$ from the top and $b$ from the bottom (so $a+b=\ell$). The lower part of the sheet, of mass $m(b/\ell)$, will be moving downward along the roof. Therefore, it will feel a friction force upward, with magnitude $\mu N = \mu m(b/\ell)g\cos\theta$. Likewise, the upper part, of mass $m(a/\ell)$, will feel a friction force downward, with magnitude $\mu m(a/\ell)g\cos\theta$.

   Because the sheet is not accelerating, the difference in these two friction forces must equal the downward force of gravity along the roof, namely $mg\sin\theta$. Therefore,

$$\begin{aligned} \mu m\left(\frac{b-a}{\ell}\right)g\cos\theta &= mg\sin\theta \\ \Longrightarrow \quad b-a &= \frac{\ell\tan\theta}{\mu}. \end{aligned} \tag{12}$$

   Note that this implies $b > a$. Also note that $b-a$ of course cannot be greater than $\ell$; therefore, if $\tan\theta > \mu$, then there are no solutions for $a$ and $b$, so the forces cannot balance, and so the sheet will accelerate down the roof. (This $\tan\theta > \mu$ result is a general result, of course, for the equilibrium of an object on an inclined plane.)

   When the object contracts, all of the above analysis holds, except that now the roles of $a$ and $b$ are reversed. The stationary point is now closer to the bottom. With $a$ and $b$ defined in the same way as above, we find (as you can verify)

$$a-b = \frac{\ell\tan\theta}{\mu}. \tag{13}$$

   Putting eqs. (12) and (13) together, we see that the stationary points of expansion ($P_{\text{e}}$) and contraction ($P_{\text{c}}$) are separated by a distance

$$d = \frac{\ell\tan\theta}{\mu}. \tag{14}$$

   During the expansion, the point $P_{\text{c}}$ moves downward a distance

$$\epsilon = \alpha d\,\Delta T = \frac{\alpha\ell\tan\theta\Delta T}{\mu}. \tag{15}$$

   and then during the contraction it remains fixed. (Equivalently, the center of the sheet moves downward by a distance of half this, for both the expansion and contraction.) Therefore, during one complete cycle (that is, during a span of 24 hours), the sheet moves downward by the distance $\epsilon$ given above.

   Plugging in the given numbers, we see that the distance the sheet moves in one year is given by

$$(365)\epsilon = \frac{(365)(17\cdot 10^{-6}(\text{C}°)^{-1})(1\,\text{m})(\tan 30°)(10°\text{C})}{1} \approx 0.036\,\text{m} = 3.6\,\text{cm}. \tag{16}$$

5. (a) The image charge lags behind the given charge by a distance $v\tau$. Therefore, from the pythagorean theorem, the separation between the two charges is $d = \sqrt{(2r)^2 + (v\tau)^2}$. The force necessary to maintain constant motion (parallel to the plate) is the negative of the Coulomb force between the charges. Hence, the desired force is

$$F = \frac{kq^2}{d^2} = \frac{kq^2}{4r^2 + v^2\tau^2} \,. \tag{17}$$

This force points at an angle of $\theta$ with respect to the normal to the plate, where $\theta$ is given by

$$\tan\theta = \frac{v\tau}{2r} \,. \tag{18}$$

(b) The component of the above force in the direction of $\mathbf{v}$ is

$$F_v \equiv F\sin\theta = \frac{kq^2}{4r^2 + v^2\tau^2}\left(\frac{v\tau}{\sqrt{4r^2 + v^2\tau^2}}\right) . \tag{19}$$

To first order in the small quantity $v\tau$, we may neglect the $v\tau$ terms in the denominator. Therefore,

$$F_v \approx \frac{kq^2 v\tau}{8r^3} \,. \tag{20}$$

This is the force necessary to overcome the damping force, $\mathbf{F} = -\gamma\mathbf{v}$. So we see that

$$\gamma = \frac{kq^2\tau}{8r^3} \,. \tag{21}$$

(c) For motion perpendicular to the plate, the lagging motion of the image charge implies that the charges will be a distance $2r + v\tau$ apart. The force between them is therefore

$$F = \frac{kq^2}{(2r + v\tau)^2} \approx \frac{kq^2}{4(r^2 + rv\tau)} \approx \frac{kq^2(r^2 - rv\tau)}{4r^4} = \frac{kq^2}{4r^2} - \frac{kq^2 v\tau}{4r^3} \,. \tag{22}$$

We see that the attractive force is slightly less than it would be if $v$ were zero. This is due to the damping force, $\mathbf{F} = -\gamma\mathbf{v}$, where

$$\gamma = \frac{kq^2\tau}{4r^3} \,. \tag{23}$$

6. Let $\ell$ and $\theta$ be the length of the string and the angle it makes with the pole, respectively, as functions of time.

The two facts we will need to solve this problem are: (1) the radial $F = ma$ equation, and (2) the conservation of energy statement.

Approximating the motion at any time by a horizontal circle (of radius $\ell\sin\theta$), we see that the vertical force applied by the string is $mg$, and hence the horizontal force is $mg\tan\theta$. Therefore, the radial $F = ma$ equation is

$$\frac{mv^2}{\ell\sin\theta} = mg\tan\theta. \tag{24}$$

Conservation of energy says that the change in KE plus the change in PE is zero. We'll write the change in KE simply as $d(mv^2/2)$ for now. We claim that the change in PE is given by $mg\ell \sin\theta \, d\theta$. This can be seen as follows.

Put a mark on the string a small distance $d\ell$ down from the contact point. After a short time, this mark will become the contact point. The height of this mark will *not* change (to first order, at least) during this process. This is true because initially the mark is a height $\ell\cos\theta$ below the initial contact point. And it is still (to first order) this far below the initial contact point when the mark becomes the contact point, because the angle is still very close to $\theta$, so any errors will be of order $d\ell\, d\theta$.

The change in height of the ball relative to this mark (whose height is essentially constant) is due to the $\ell - d\ell$ length of string in the air "swinging" up through an angle $d\theta$. Multiplying by $\sin\theta$ to obtain the vertical component of this arc, we see that the change in height is $((\ell - d\ell)d\theta)\sin\theta$. This equals $\ell\sin\theta\, d\theta$, to first order, as was to be shown.

Therefore, conservation of energy gives

$$\frac{1}{2}d(mv^2) + mg\ell\sin\theta\, d\theta = 0. \tag{25}$$

We will now use eqs. (24) and (25) to solve for $\ell$ in terms of $\theta$. Substituting the $v^2$ from eq. (24) into eq. (25) gives

$$\begin{aligned}
& d(\ell\sin\theta\tan\theta) + 2\ell\sin\theta\, d\theta = 0 \\
\Longrightarrow\quad & (d\ell\sin\theta\tan\theta + \ell\cos\theta\tan\theta\, d\theta + \ell\sin\theta\sec^2\theta) + 2\ell\sin\theta\, d\theta = 0 \\
\Longrightarrow\quad & d\ell\frac{\sin^2\theta}{\cos\theta} + 3\ell\sin\theta\, d\theta + \ell\frac{\sin\theta}{\cos^2\theta} = 0 \\
\Longrightarrow\quad & \int\frac{d\ell}{\ell} = -\int\frac{3\cos\theta\, d\theta}{\sin\theta} - \int\frac{d\theta}{\sin\theta\cos\theta} \\
\Longrightarrow\quad & \ln\ell = -3\ln(\sin\theta) + \ln\left(\frac{\cos\theta}{\sin\theta}\right) + C \\
\Longrightarrow\quad & \ell = A\frac{\cos\theta}{\sin^4\theta}, \qquad \text{where} \quad A = L\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)
\end{aligned} \tag{26}$$

is determined from the initial condition, $\ell = L$ when $\theta = \theta_0$. Note that this result implies that $\theta = \pi/2$ when the ball hits the pole (that is, when $\ell = 0$). The last integral in the fourth line above can be found in various ways. One is to multiply by $\cos\theta/\cos\theta$, and then note that $d\theta/\cos^2\theta = d(\tan\theta)$.

Now let's find the position where the ball hits the pole. The vertical distance a small piece of the string covers is $dy = d\ell\,\cos\theta$. So the ball hits the pole at a $y$ value (relative to the top) given by

$$y = \int d\ell\,\cos\theta = A\int d\left(\frac{\cos\theta}{\sin^4\theta}\right)\cos\theta, \tag{27}$$

where the integral runs from $\theta_0$ to $\pi/2$, and $A$ is given in eq. (26). We may now integrate by parts to obtain

$$\frac{y}{A} \;=\; \left(\frac{\cos\theta}{\sin^4\theta}\right)\cos\theta - \int\left(\frac{\cos\theta}{\sin^4\theta}\right)(-\sin\theta)\, d\theta$$

$$
\begin{aligned}
&= \frac{\cos^2\theta}{\sin^4\theta} + \int \frac{\cos\theta}{\sin^3\theta}\,d\theta \\
&= \left(\frac{\cos^2\theta}{\sin^4\theta} - \frac{1}{2\sin^2\theta}\right)\Bigg|_{\theta_0}^{\pi/2} . \qquad (28)
\end{aligned}
$$

Using the value of $A$ given in eq. (26), we obtain

$$
\begin{aligned}
y &= L\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)\left(-\frac{1}{2} - \left(\frac{\cos^2\theta_0}{\sin^4\theta_0} - \frac{1}{2\sin^2\theta_0}\right)\right) \\
&= L\left(\frac{\sin^4\theta_0}{\cos\theta_0}\right)\left(-\frac{\cos^2\theta_0}{\sin^4\theta_0} + \frac{\cos^2\theta_0}{2\sin^2\theta_0}\right) \\
&= -L\cos\theta_0\left(1 - \frac{\sin^2\theta_0}{2}\right). \qquad (29)
\end{aligned}
$$

Since the ball starts at a position $y = -L\cos\theta_0$, we see that it rises up a distance $\Delta y = (1/2)L\cos\theta_0\sin^2\theta_0$ during the course of its motion. (This change in height happens to be maximum when $\tan\theta_0 = \sqrt{2}$, in which case $\Delta y = L/3\sqrt{3}$.)

By conservation of energy, we can find the final speed from

$$
\frac{1}{2}mv_f^2 = \frac{1}{2}mv_i^2 - mg\left(\frac{1}{2}L\cos\theta_0\sin^2\theta_0\right) \qquad (30)
$$

From eq. (24), we have

$$
v_i^2 = gL\frac{\sin^2\theta_0}{\cos\theta_0} . \qquad (31)
$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}mv_f^2 &= \frac{1}{2}mgL\frac{\sin^2\theta_0}{\cos\theta_0} - \frac{1}{2}mgL\cos\theta_0\sin^2\theta_0 \\
&= \frac{1}{2}mgL\sin^2\theta_0\left(\frac{1}{\cos\theta_0} - \cos\theta_0\right) \\
&= \frac{1}{2}mgL\frac{\sin^4\theta_0}{\cos\theta_0} . \qquad (32)
\end{aligned}
$$

Hence,

$$
v_f^2 = gL\frac{\sin^4\theta_0}{\cos\theta_0} . \qquad (33)
$$

Combining eqs. (31) and (33), we finally have

$$
\frac{v_f}{v_i} = \sin\theta_0. \qquad (34)
$$