Solutions

10th Annual
Boston Area Undergraduate
Physics Competition

April 24, 2004

1. (a) From best to worst, the ordering of the strategies is (ii), (iii), (i). We can demonstrate this by using conservation of momentum. There are no external horizontal forces on the sled and the snow, so the total momentum of the sled plus the snow is constant in time.

   Strategy (ii) therefore beats strategy (iii), because the snow in (ii) ends up with no forward momentum, while the snow in (iii) continues to move forward with the sled. The snow in (ii) therefore has less momentum than the snow in (iii), so the sled in (ii) must have more momentum than the sled in (iii).

   Strategy (iii) beats strategy (i) for the following reason. When a snowflake is brushed off the sled in strategy (i), it initially has the same forward speed as the sled as they both sail across the frictionless ice. But when the next snowflake hits the sled, the sled slows down. The brushed-off snowflake therefore now has a larger forward speed than the sled. The sled therefore moves at a speed that is slower than the speed of the center of mass of the sled-plus-snowflake system. But this latter speed is simply the speed of the sled in (iii).

   (b) Swinging your arms does indeed help. Consider the angular momentum of your body relative to your feet. The friction force at your feet provides no torque relative to your feet, so the only external torque is the torque due to gravity (which is what is making you fall over). However, for a small enough period of time, this torque won't angularly accelerate you much, so your angular momentum with respect to your feet is approximately constant.

   Now assume that you start swinging your arms around with the orientation such that your hands are moving forward at the lowest point and backward at the highest point. The right-hand rule then says that your arms have angular momentum which points to your right. But since your angular momentum is approximately constant, there must now be something that has angular momentum pointing to your left. This something is you. You will therefore rotate "forwards" relative to your feet. In other words, you won't fall backwards (assuming that you swing your arms around fast enough).

   Note that it is the *change* in the angular momentum of your arms that is relevant. In other words, the swinging only helps you at the start. Once your arms reach their maximum speed (which in practice happens very quickly), the swinging doesn't help you anymore. But hopefully you've managed to get your center of mass back up above your feet by this time.

2. Let the total mass of the rope be m, and let a fraction f of it hang in the air. Consider the right half of this section. Its weight, $(f/2)mg$, must be balanced by the vertical component, $T\sin\theta$, of the tension at the point where it joins the part of the rope touching the right platform. The tension at that point is therefore $T = (f/2)mg/\sin\theta$.

   Now consider the part of the rope touching the right platform, which has mass $(1-f)m/2$. The normal force from the platform is $N = (1-f)(mg/2)\cos\theta$, so the maximal friction force also equals $(1-f)(mg/2)\cos\theta$, because $\mu = 1$. This fiction force must balance the sum of the gravitational force component along the plane, which is $(1-f)(mg/2)\sin\theta$, plus the tension at the lower end, which we found above. Therefore,

$$\frac{1}{2}(1-f)mg\cos\theta = \frac{1}{2}(1-f)mg\sin\theta + \frac{fmg}{2\sin\theta}\,. \tag{1}$$

   This gives

$$f = \frac{F(\theta)}{1+F(\theta)}\,, \qquad \text{where} \quad F(\theta) \equiv \cos\theta\sin\theta - \sin^2\theta. \tag{2}$$

   This expression for f is a monotonically increasing function of $F(\theta)$, as you can check. The maximal f is therefore obtained when $F(\theta)$ is as large as possible. Using the double-angle formulas, we can rewrite $F(\theta)$ as

$$F(\theta) = \frac{1}{2}(\sin 2\theta + \cos 2\theta - 1). \tag{3}$$

   The derivative of this is $\cos 2\theta - \sin 2\theta$, which equals zero when $\tan 2\theta = 1$. Therefore,

$$\theta_{\text{max}} = 22.5^\circ. \tag{4}$$

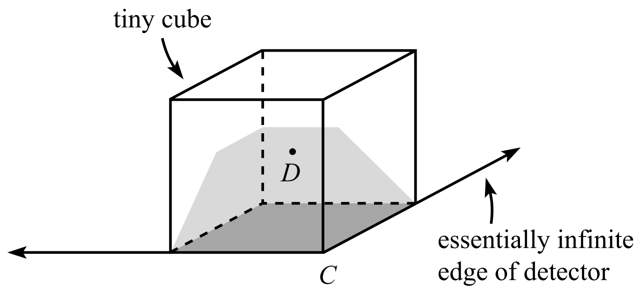
   Eq. (3) then yields $F(\theta_{\text{max}}) = (\sqrt{2}-1)/2$, and so eq. (2) gives

$$f_{\text{max}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2} \approx 0.172. \tag{5}$$

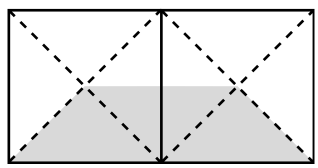
3. Let's consider point B first. This point is the center of a cube of side length d, one of the faces of which is the detector. Since the radiation from the particle is isotropic, 1/6 of it passes through each face of the cube. Therefore, 1/6 of the particle's radiation is detected by the square when the particle is at point B.

   Now consider point A. This point is the center of a cube of side length $2d$. The detector spans one quarter of one of these faces. Combining this fact with the above reasoning tells us that $(1/4)(1/6) = 1/24$ of the particle's radiation is detected by the square when the particle is at point A.

   Lastly, consider a point (call it D) very close to C. This case is a little tricker. Point D is the center of a tiny cube which has as its bottom face a tiny square at the corner of the detector. What areas on this cube correspond to radiation hitting the detector? From the above reasoning, 1/6 of the particle's radiation passes through the bottom face. The other relevant areas on the cube are shown as the lighter shaded regions below.

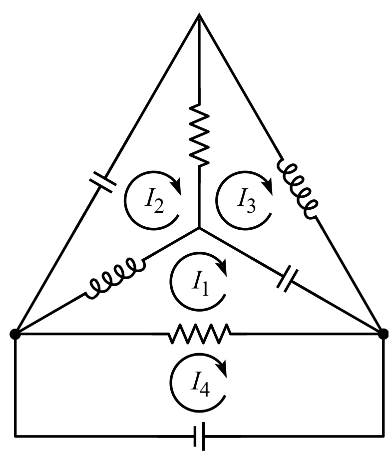


The top horizontal boundary of the lightly shaded region corresponds to the two far edges of the detector, which are essentially infinitely far away. The two diagonal boundaries correspond to the two edges of the detector that emanate from point C.[1] The lightly shaded region covers $3/8$ of the area of the two side faces, as can be seen by flattening out these faces, as shown.



The eight triangles shown in this figure have equal amounts of radiation hitting them, so the shaded $3/8$ of the area corresponds to $3/8$ of the radiation passing through the faces. These two faces represent $1/3$ of the cube, so the total fraction of the particle's radiation that hits the detector is $1/6 + (3/8)(1/3) = 7/24$.

4. We may as well consider the tetrahedron to be a planar circuit, as shown in the diagram below (which looks just like the original 3D diagram). Let the four loop currents be as shown.



[1]These diagonal boundaries are indeed straight lines, which can be seen by noting that each of them is determined by the intersection of two planes, one of which is a face of the tiny cube, and the other of which is the plane determined by an edge of the detector (emanating from point C) and point D.

Since $\omega = 1/\sqrt{LC}$ and $R = \sqrt{L/C}$, the impedances associated with the resistors, inductors, and capacitors take the form,

$$\begin{aligned} Z_R &= R, \\ Z_L &= i\omega L = i\sqrt{\frac{L}{C}} = iR, \\ Z_C &= \frac{-i}{\omega C} = -i\sqrt{\frac{L}{C}} = -iR. \end{aligned} \tag{6}$$

The four loop equations expressing the fact that the voltage drop around a loop is zero are then

$$\begin{aligned} (I_1 - I_4)R + (I_1 - I_2)(iR) + (I_1 - I_3)(-iR) &= 0, \\ I_2(-iR) + (I_2 - I_3)R + (I_2 - I_1)(iR) &= 0, \\ I_3(iR) + (I_3 - I_1)(-iR) + (I_3 - I_2)R &= 0, \\ (I_4 - I_1)R &= V_0. \end{aligned} \tag{7}$$

These simplify to

$$\begin{aligned} (I_1 - I_4) + i(I_3 - I_2) &= 0, \\ (I_2 - I_3) - iI_1 &= 0, \\ (I_3 - I_2) + iI_1 &= 0, \\ (I_4 - I_1) &= V_0/R. \end{aligned} \tag{8}$$

The second and third equations are equivalent, so we in fact have only three equations for our four unknown currents (more on this in the remark below). Multiplying the second equation by i and adding it to the first gives $2I_1 - I_4 = 0 \Longrightarrow I_1 = I_4/2$. Plugging this into the last equation then gives the amplitude of total current through the circuit as

$$I_4 = \frac{2V_0}{R}\,. \tag{9}$$

The effective impedance of the entire circuit is therefore $R/2$, so we see that the upper five lines in the figure effectively act like a resistor of resistance R in parallel with the bottom resistor R.

REMARK: The above four equations determine the difference $I_2 - I_3$ to be iV_0/R, but they don't determine I_2 and I_3 individually. These two currents can indeed take on any values, as long as their difference is iV_0/R. Any equal increase in their values simply corresponds to dumping more current on the union of the top two loops (the "2" and "3" loops), which consists of two inductors and two capacitors at resonance (because $\omega = 1/\sqrt{LC}$).

5. Assume that the particle slides off to the right. Let v_x and v_y be its horizontal and vertical velocities, with rightward and downward taken to be positive, respectively. Let V_x be the velocity of the hemisphere, with leftward taken to be positive. Conservation of momentum gives

$$mv_x = MV_x \qquad \Longrightarrow \qquad V_x = \left(\frac{m}{M}\right) v_x. \tag{10}$$

Consider the moment when the particle is located at an angle θ down from the top of the hemisphere. Locally, it is essentially on a plane inclined at angle θ, so the three velocity components are related by

$$\frac{v_y}{v_x + V_x} = \tan\theta \qquad \Longrightarrow \qquad v_y = \tan\theta\left(1 + \frac{m}{M}\right)v_x. \tag{11}$$

To see why this is true, look at things in the frame of the hemisphere. In that frame, the particle moves to the right at speed $v_x + V_x$, and downward at speed v_y. Eq. (11) represents the constraint that the particle remains on the hemisphere, which is inclined at an angle θ at the given location.

Let us now apply conservation of energy. In terms of θ, the particle has fallen a distance $R(1-\cos\theta)$, so conservation of energy gives

$$\frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}MV_x^2 = mgR(1-\cos\theta). \tag{12}$$

Using eqs. (10) and (11), we can solve for v_x^2 to obtain

$$v_x^2 = \frac{2gR(1-\cos\theta)}{(1+r)\Big(1+(1+r)\tan^2\theta\Big)}, \qquad \text{where} \;\; r \equiv \frac{m}{M}. \tag{13}$$

This function of θ starts at zero for $\theta = 0$ and increases as θ increases. It then achieves a maximum value before heading back down to zero at $\theta = \pi/2$. However, v_x *cannot* actually decrease, because there is no force available to pull the particle to the left. So what happens is that v_x initially increases due to the non-zero normal force that exists while contact remains. But then v_x reaches its maximum, which corresponds to the normal force going to zero and the particle losing contact with the hemisphere. The particle then sails through the air with constant v_x. Our goal, then, is to find the angle θ for which the v_x^2 in eq. (13) is maximum. Setting the derivative equal to zero gives

$$\begin{aligned}
0 &= \Big(1+(1+r)\tan^2\theta\Big)\sin\theta - (1-\cos\theta)(1+r)\frac{2\tan\theta}{\cos^2\theta} \\
\Longrightarrow \quad 0 &= \Big(1+(1+r)\tan^2\theta\Big)\cos^3\theta - 2(1+r)(1-\cos\theta) \\
\Longrightarrow \quad 0 &= \cos^3\theta + (1+r)(\cos\theta - \cos^3\theta) - 2(1+r)(1-\cos\theta) \\
\Longrightarrow \quad 0 &= r\cos^3\theta - 3(1+r)\cos\theta + 2(1+r). \qquad (14)
\end{aligned}$$

This is the desired equation that determines θ. It is a cubic equation, so in general it can't be solved so easily for θ. But in the special case of $r = 1$, we have

$$0 = \cos^3\theta - 6\cos\theta + 4. \tag{15}$$

By inspection, $\cos\theta = 2$ is an (unphysical) solution, so we find

$$(\cos\theta - 2)(\cos^2\theta + 2\cos\theta - 2) = 0. \tag{16}$$

The physical root of the quadratic equation is

$$\cos\theta = \sqrt{3} - 1 \approx 0.732 \qquad \Longrightarrow \qquad \theta \approx 42.9^\circ. \tag{17}$$

Alternate solution: In the reference frame of the hemisphere, the horizontal speed of the particle $v_x + V_y = (1+r)v_x$. The total speed in this frame equals this horizontal speed divided by $\cos\theta$, so

$$v = \frac{(1+r)v_x}{\cos\theta}\,. \tag{18}$$

The particle leaves the hemisphere when the normal force goes to zero. The radial $F = ma$ equation therefore gives

$$mg\cos\theta = \frac{mv^2}{R}\,. \tag{19}$$

You might be concerned that we have neglected the sideways fictitious force in the accelerating frame of the hemisphere. However, the hemisphere is *not* accelerating beginning at the moment when the particle loses contact, because the normal force has gone to zero. Therefore, eq. (19) looks exactly like it does for the familiar problem involving a fixed hemisphere; the difference in the two problems is in the calculation of v.

Using eqs. (13) and (18) in eq. (19) gives

$$mg\cos\theta = \frac{m(1+r)^2}{R\cos^2\theta}\cdot\frac{2gR(1-\cos\theta)}{(1+r)\Big(1+(1+r)\tan^2\theta\Big)}\,. \tag{20}$$

Simplifying this yields

$$\Big(1+(1+r)\tan^2\theta\Big)\cos^3\theta = 2(1+r)(1-\cos\theta), \tag{21}$$

which is the same as the second line in eq. (14). The solution proceeds as above.

REMARK: Let's look at a few special cases of the $r \equiv m/M$ value. In the limit $r \to 0$ (in other words, the hemisphere is essentially bolted down), eq. (14) gives

$$\cos\theta = 2/3 \qquad \Longrightarrow \qquad \theta \approx 48.2^\circ, \tag{22}$$

a result which may look familiar to you. In the limit $r \to \infty$, eq. (14) reduces to

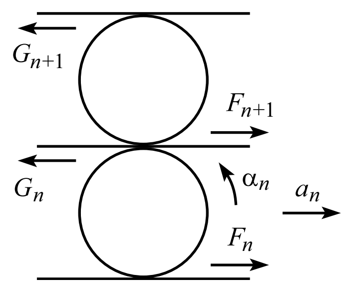
$$0 = \cos^3\theta - 3\cos\theta + 2 \qquad \Longrightarrow \qquad 0 = (\cos\theta - 1)^2(\cos\theta + 2). \tag{23}$$

Therefore, $\theta = 0$. In other words, the hemisphere immediately gets squeezed out very fast to the left.

For other values of r, we can solve eq. (14) either by using the formula for the roots of a cubic equation (very messy), or by simply doing things numerically. A few numerical results are:

r	$\cos\theta$	θ
0	.667	48.2°
1/2	.706	45.1°
1	.732	42.9°
2	.767	39.9°
10	.858	30.9°
100	.947	18.8°
1000	.982	10.8°
∞	1	0°

6. Both cylinders in a given row move in the same manner, so we may simply treat them as one cylinder with mass $m = 2M$. Let the forces that the boards exert on the cylinders be labelled as shown. "F" is the force from the plank below a given cylinder, and "G" is the force from the plank above it.



Note that by Newton's third law, we have $F_{n+1} = G_n$, because the planks are massless.

Our strategy will be to solve for the linear and angular accelerations of each cylinder in terms of the accelerations of the cylinder below it. Since we want to solve for two quantities, we will need to produce two equations relating the accelerations of two successive cylinders. One equation will come from a combination of $F = ma$, $\tau = I\alpha$, and Newton's third law. The other will come from the nonslipping condition.

With the positive directions for a and α defined as in the figure, $F = ma$ on the nth cylinder gives

$$F_n - G_n = ma_n, \tag{24}$$

and $\tau = I\alpha$ on the nth cylinder gives

$$(F_n + G_n)R = \frac{1}{2}mR^2\alpha_n \qquad \Longrightarrow \qquad F_n + G_n = \frac{1}{2}mR\alpha_n \tag{25}$$

Solving the previous two equations for F_n and G_n gives

$$\begin{aligned} F_n &= \frac{1}{2}\left(ma_n + \frac{1}{2}mR\alpha_n\right), \\ G_n &= \frac{1}{2}\left(-ma_n + \frac{1}{2}mR\alpha_n\right). \end{aligned} \tag{26}$$

But we know that $F_{n+1} = G_n$. Therefore,

$$a_{n+1} + \frac{1}{2}R\alpha_{n+1} = -a_n + \frac{1}{2}R\alpha_n. \tag{27}$$

We will now use the fact that the cylinders don't slip with respect to the boards. The acceleration of the board above the nth cylinder is $a_n - R\alpha_n$. But the acceleration of this same board, viewed as the board below the $(n+1)$st cylinder, is $a_{n+1} + R\alpha_{n+1}$. Therefore,

$$a_{n+1} + R\alpha_{n+1} = a_n - R\alpha_n. \tag{28}$$

Eqs. (27) and (28) are a system of two equations in the two unknowns, a_{n+1} and α_{n+1}, in terms of a_n and α_n. Solving for a_{n+1} and α_{n+1} gives

$$\begin{aligned} a_{n+1} &= -3a_n + 2R\alpha_n, \\ R\alpha_{n+1} &= 4a_n - 3R\alpha_n. \end{aligned} \tag{29}$$

We can write this in matrix form as

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{30}$$

We therefore have

$$\begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ R\alpha_1 \end{pmatrix}. \tag{31}$$

Consider now the eigenvectors and eigenvalues of the above matrix. The eigenvectors are found via[2]

$$\begin{vmatrix} -3-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} = 0 \qquad \Longrightarrow \qquad \lambda_\pm = -3 \pm 2\sqrt{2}. \tag{32}$$

The eigenvectors are then

$$\begin{aligned} V_+ &= \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_+ = -3 + 2\sqrt{2}, \\ V_- &= \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \qquad \text{for } \lambda_- = -3 - 2\sqrt{2}. \end{aligned} \tag{33}$$

Note that $|\lambda_-| > 1$, so $\lambda_-^n \to \infty$ as $n \to \infty$. This means that if the initial $(a_1, R\alpha_1)$ vector has any component in the V_- direction, then the $(a_n, R\alpha_n)$ vectors will head to infinity. This violates conservation of energy. Therefore, the $(a_1, R\alpha_1)$ vector must be proportional to V_+.[3] That is, $R\alpha_1 = \sqrt{2}a_1$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$a = a_1 + \sqrt{2}a_1 \qquad \Longrightarrow \qquad a_1 = \frac{a}{\sqrt{2}+1} = (\sqrt{2}-1)a. \tag{34}$$

Remark: Let us consider the general case where the cylinders have a moment of inertia of the form $I = \beta MR^2$. Using the above arguments, you can show that eq. (30) becomes

$$\begin{pmatrix} a_{n+1} \\ R\alpha_{n+1} \end{pmatrix} = \frac{1}{1-\beta} \begin{pmatrix} -(1+\beta) & 2\beta \\ 2 & -(1+\beta) \end{pmatrix} \begin{pmatrix} a_n \\ R\alpha_n \end{pmatrix}. \tag{35}$$

[2] λ_+ happens to be the negative of the $f_{\max}$ result found in Problem 2. An interesting fact, but also a completely random one, I believe.

[3] This then means that the $(a_n, R\alpha_n)$ vectors head to zero as $n \to \infty$, because $|\lambda_+| < 1$. Also, note that the accelerations change sign from one level to the next, because λ_+ is negative.

And you can show that the eigenvectors and eigenvalues are

$$
\begin{aligned}
V_+ &= \begin{pmatrix} \sqrt{\beta} \\ 1 \end{pmatrix}, \qquad \text{for } \lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}, \\
V_- &= \begin{pmatrix} \sqrt{\beta} \\ -1 \end{pmatrix}, \qquad \text{for } \lambda_- = \frac{\sqrt{\beta}+1}{\sqrt{\beta}-1}.
\end{aligned} \tag{36}
$$

As above, we cannot have the exponentially growing solution, so we must have only the V_+ solution. We therefore have $R\alpha_1 = a_1/\sqrt{\beta}$. Combining this with the fact that the given acceleration, a, of the bottom board equals $a_1 + R\alpha_1$, we obtain

$$
a = a_1 + \frac{a_1}{\sqrt{\beta}} \qquad \Longrightarrow \qquad a_1 = \left(\frac{\sqrt{\beta}}{1+\sqrt{\beta}}\right) a. \tag{37}
$$

You can verify that all of these results agree with the $\beta = 1/2$ results obtained above.

Let's now consider a few special cases of the

$$
\lambda_+ = \frac{\sqrt{\beta}-1}{\sqrt{\beta}+1} \tag{38}
$$

eigenvalue, which gives the ratio of the accelerations in any level to the ones in the next level down.

- If $\beta = 0$ (all the mass of a cylinder is located at the center), then we have $\lambda_+ = -1$. In other words, the accelerations have the same magnitudes but different signs from one level to the next. The cylinders simply spin in place while their centers remain fixed. The centers are indeed fixed, because $a_1 = 0$, from eq. (37).
- If $\beta = 1$ (all the mass of a cylinder is located on the rim), then we have $\lambda_+ = 0$. In other words, there is no motion above the first level. The lowest cylinder basically rolls on the bottom side of the (stationary) plank right above it. Its acceleration is $a_1 = a/2$, from eq. (37).
- If $\beta \to \infty$ (the cylinders have long massive extensions that extend far out beyond the rim), then we have $\lambda_+ = 1$. In other words, all the levels have equal accelerations. This fact, combined with the $R\alpha_1 = a_1/\sqrt{\beta} \approx 0$ result, shows that there is no rotational motion at any level, and the whole system simply moves to the right as a rigid object with acceleration $a_1 = a$, from eq. (37).