Boston Area Undergraduate
Physics Competition
April 22, 2000

SOLUTIONS

1. Let x be the distance from P to the center of the stick. The moment of inertia of the stick (around the center of mass) is

$$I_{\rm CM} = \int_{-L/2}^{L/2} x^2\, dm = \int_{-L/2}^{L/2} x^2 \sigma\, dx = \frac{(\sigma L)L^2}{12} = \frac{ML^2}{12}\,. \qquad (1)$$

Using the parallel-axis theorem, the moment of inertia around point P is

$$I_P = \frac{ML^2}{12} + Mx^2. \qquad (2)$$

The torque, relative to P, is due to gravity effectively acting at the center of mass. Therefore, when the stick makes an angle θ with respect to the horizontal, the torque is $\tau = Mgx\cos\theta$. Hence, $\tau = I_P\alpha$ gives

$$\alpha = \frac{Mgx\cos\theta}{\frac{ML^2}{12} + Mx^2}\,. \qquad (3)$$

The stick will fall quickest when the coefficient of $\cos\theta$ is maximum. Taking the derivative with respect to x, we find that α is maximized when

$$x = \frac{L}{\sqrt{12}}\,. \qquad (4)$$

2. (a) Add three image charges to create the square of charges shown below. It is easy to see that the total electric field caused by all four charges is perpendicular to the two planes. Since this field satisfies the boundary conditions required by the conducting planes, it must be the same (due to the uniqueness theorem) as the field produced by the actual charge q and the negative charges that build up on the planes.

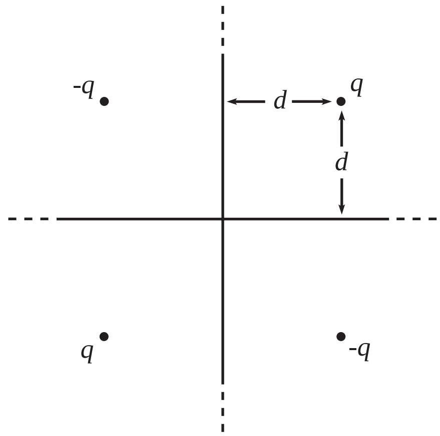
The total potential energy of the entire system of charges is

$$V_{\rm tot} = 4\left(\frac{-q^2}{2d}\right) + 2\left(\frac{q^2}{2\sqrt{2}\,d}\right). \qquad (5)$$

The potential energy of the actual charge q and the negative charges that build up on the planes is $V_{\rm tot}/4$. (This can be seen by noting that the energy of a system of charges is equal to the integral of the square of the electric field. And the actual setup has an electric field in only one quadrant of space.) The work $W_{\rm in}$ is equal to the potential energy of the actual system of charges. Therefore,

$$W_{\rm in} = \frac{V_{\rm tot}}{4} = \frac{q^2}{d}\left(-\frac{1}{2} + \frac{1}{4\sqrt{2}}\right), \qquad (6)$$

which is negative, as it should be.



(b) When the charge q is moved away from the insulating planes, it feels a force as if it is interacting with the three image charges fixed at their positions. (This is true because at any point in the upper right quadrant, the electric field due to the image charges must be precisely equal to the electric field due to the charges on the insulating plates. This is why we picked these image charges, after all.)

The work W_{out} is thus equal to the work done in separating the given charge from the three image charges. This work is

$$\begin{aligned} W_{\text{out}} &= 2\left(\frac{q^2}{2d}\right) - \left(\frac{q^2}{2\sqrt{2}\,d}\right) \\ &= \frac{q^2}{d}\left(1 - \frac{1}{2\sqrt{2}}\right), \end{aligned} \tag{7}$$

which is positive, as it should be.

(c) The potential energy of the system of charges on the insulating planes is equal to the total work done on the system, which is

$$W_{\text{in}} + W_{\text{out}} = \frac{q^2}{d}\left(\frac{1}{2} - \frac{1}{4\sqrt{2}}\right), \tag{8}$$

which is positive, as it should be.

3. The condition that light is able to take a circular path around the planet is that this circular path takes the least amount of time, compared to all nearby paths. (This is Fermat's principle of least time.) This condition then implies that nearby circular paths take the same amount of time (to first order in their size difference, at least).

The speed of light in a medium is proportional to the reciprocal of the index of refraction. Therefore, the condition for the existence of a circular path is[1]

$$\frac{n(R+h)}{n(R)} = \frac{R}{R+h}\,. \tag{9}$$

Expanding this to first order in h gives $R = -n/(dn/dh)$. But the given information $n = 1 + \epsilon\rho$ says that $dn/dh = \epsilon\, d\rho/dh$. Therefore,

$$R = -\frac{n}{\epsilon\, d\rho/dh}\,. \tag{10}$$

We must therefore find $d\rho/dh$. We will need three facts.

(a) The first is

$$\frac{d\rho}{dh} = \frac{\rho_E}{P_E}\frac{dP}{dh}\,. \tag{11}$$

This follows from the ideal gas law, $PV = nRT$. Dividing through by V, we find $P \propto \rho$. Since the temperature of the planet's atmosphere is independent of height, the constant of proportionality is independent of height. And since the temperature is the same as on the earth's surface, we have $\rho/P = \rho_E/P_E$, from which eq. (11) follows.

(b) The second is

$$\frac{dP}{dh} = -g\rho. \tag{12}$$

This follows from the usual consideration of a small column of air (with height dh and base area A), which has a net force on it equal to $-A(dP/dh)\,dh - (\rho A\, dh)g$. Setting this equal to zero yields eq. (12).

(c) The third is

$$g = \frac{g_E}{R_E}R. \tag{13}$$

This follows from the definition $g \equiv GM/R^2$, along with the assumption that the densities of the earth and the planet are equal. Writing M as the density times $4\pi R^3/3$ gives $g \propto R$, from which eq. (13) follows.

These three facts imply

$$\frac{d\rho}{dh} = -\left(\frac{\rho_E}{P_E}\right)\left(\frac{g_E}{R_E}R\right)\rho. \tag{14}$$

Plugging this into eq. (10) gives

$$R = \frac{nR_E P_E}{\epsilon g_E \rho_E \rho R}\,. \tag{15}$$

[1] You can also derive this result by considering the atmosphere to consist of layers with different n's, and demanding that there be total internal reflection in a given layer.

Using $n = 1 + \epsilon\rho$, and also the assumption $\rho = \rho_E$, and then solving for R gives

$$R = \sqrt{\frac{(1+\epsilon\rho_E)R_E P_E}{\epsilon g_E \rho_E^2}}\,. \tag{16}$$

REMARK: The quantity $\epsilon\rho_E$ is roughly equal to $3 \cdot 10^{-4}$. Using this value (its effect in the numerator of eq. (16) is negligible), along with $R_E \approx 6.4 \cdot 10^6\,\mathrm{m}$, $g_E \approx 10\,\mathrm{m/s^2}$, $P_E \approx 1 \cdot 10^5\,\mathrm{kg/ms^2}$, and $\rho_E \approx 1.2\,\mathrm{kg/m^3}$, gives

$$R \approx 1.3 \cdot 10^7\,\mathrm{m} \approx 2R_E. \;\blacksquare \tag{17}$$

4. (a) Let $\mu \equiv M/N$ be the mass of each ball in the semicircle. We want the deflection angle in each collision to be $\theta = \pi/N$. However, if the ratio μ/m is too small, then this angle of deflection is not possible. Let's be quantitative about this.

Lemma: If a mass m collides with a stationary mass μ, then the maximum angle of deflection is given by

$$\sin\theta_{\text{max}} = \frac{\mu}{m}. \tag{18}$$

Proof: Let V be the initial speed of mass m in the lab frame. Then in the CM frame, it is easy to see that m and μ head toward each other with initial speeds

$$v_m = \frac{\mu V}{m+\mu}, \qquad \text{and} \qquad v_\mu = \frac{mV}{m+\mu}\,. \tag{19}$$

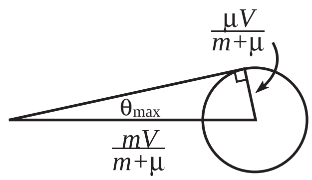
Because the collision is elastic, these speeds are also the final speeds in the CM frame, independent of the final directions of motion (which are of course opposites of each other.)

To return to the lab frame, we must add on the sideways speed of $v_\mu = mV/(m+\mu)$ to each particle. Therefore, the final velocity of m in the lab frame is the vector sum,

$$\mathbf{V}_f = \left(\frac{mV}{m+\mu}\right)\hat{\mathbf{x}} + \left(\frac{\mu V}{m+\mu}\right)\hat{\mathbf{r}}, \tag{20}$$

where $\hat{\mathbf{r}}$ is the unit vector representing the direction on the CM frame. To maximize the angle of deflection in the lab frame, we clearly want to have the situation show below, where the total velocity vector is tangent to the circle. In this case, we have $\sin\theta_{\text{max}} = \mu/m$, as was to be shown.[2]

[2] You can also derive this result by going through the straightforward (but tedious) calculation using conservation of energy and momentum during the collision. You will find that a solution exists (for, say, the final speed of m) only if $\sin\theta \le \mu/m$.



(This result holds for $\mu < m$. If $\mu > m$, then the maximum angle of deflection is $180°$.) ■

In the problem at hand, we have $\theta = \pi/N$. Since θ is small, we may use $\sin\theta \approx \theta$ to write the $\sin\theta \leq \mu/m$ condition as

$$\theta \leq \frac{\mu}{m} \qquad \Longrightarrow \qquad \frac{\pi}{N} \leq \frac{(M/N)}{m} \qquad \Longrightarrow \qquad \pi \leq \frac{M}{m}. \tag{21}$$

(Given m, it is clear that there must be a lower bound on M, because if M is very small, the mass m will simply plow through the semicircle.)

(b) From the above figure, the final speed after one bounce, in the case of maximum deflection angle, is

$$V_f = V\frac{\sqrt{m^2 - \mu^2}}{m+\mu} \approx V\left(1 - \frac{\mu}{m}\right), \tag{22}$$

to first order in the small quantity μ/m. The same reasoning holds for all bounces, so the speed decreases by the factor $(1 - \mu/m)$ after each bounce. In the special case where $\mu/m = \pi/N$, we see that after the N bounces off all the balls, the ratio of m's final speed to initial speed is

$$\frac{V_{\text{final}}}{V_{\text{initial}}} = \left(1 - \frac{\pi}{N}\right)^N \approx e^{-\pi}. \tag{23}$$

It doesn't get any nicer than that!

($e^{-\pi}$ is roughly equal to 1/23, so only about 4% of the initial speed remains.)

5. The angular velocity of the turntable is $\Omega\hat{\mathbf{z}}$. Let the angular velocity of the ball be $\boldsymbol{\omega}$. If the ball is at position $\mathbf{r}$ (with respect to the lab frame), then its velocity (with respect to the lab frame) may be broken up into the velocity of the turntable (at position $\mathbf{r}$) plus the ball's velocity relative to the turntable. The non-slipping condition says that this latter velocity is given by $\boldsymbol{\omega} \times (a\hat{\mathbf{z}})$. (We'll use "$a$" to denote the radius of the sphere.) The ball's velocity with respect to the lab frame is thus

$$\mathbf{v} = (\Omega\hat{\mathbf{z}}) \times \mathbf{r} + \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{24}$$

The angular momentum of the ball is

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{25}$$

The friction force from the ground is responsible for changing both the momentum and the angular momentum of the ball. $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,, \tag{26}$$

and $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = \frac{d\mathbf{L}}{dt}\,, \tag{27}$$

since the force is applied at position $-a\hat{\mathbf{z}}$ relative to the ball's center.

We will now use the previous four equations to demonstrate that the ball undergoes circular motion. Our goal will be to produce an equation of the form $d\mathbf{v}/dt = \Omega'\hat{\mathbf{z}} \times \mathbf{v}$, since this describes circular motion, with frequency Ω' (to be determined).

Plugging the expressions for $\mathbf{L}$ and $\mathbf{F}$ from eqs. (25) and (26) into eq. (27) gives

$$\begin{aligned} (-a\hat{\mathbf{z}}) \times \left(m\frac{d\mathbf{v}}{dt}\right) &= I\frac{d\boldsymbol{\omega}}{dt} \\ \Longrightarrow \qquad \frac{d\boldsymbol{\omega}}{dt} &= -\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\,. \end{aligned} \tag{28}$$

Taking the derivative of eq. (24) then gives

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times (a\hat{\mathbf{z}}) \\ &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\left(\frac{am}{I}\right)\hat{\mathbf{z}} \times \frac{d\mathbf{v}}{dt}\right) \times (a\hat{\mathbf{z}}). \end{aligned} \tag{29}$$

Since the vector $d\mathbf{v}/dt$ lies in the horizontal plane, it is easy to work out the cross-product in the right term (or just use the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$) to obtain

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}} \times \mathbf{v} - \left(\frac{a^2 m}{I}\right)\frac{d\mathbf{v}}{dt} \\ \Longrightarrow \qquad \frac{d\mathbf{v}}{dt} &= \left(\frac{\Omega}{1 + (a^2 m/I)}\right)\hat{\mathbf{z}} \times \mathbf{v}. \end{aligned} \tag{30}$$

For a uniform sphere, $I = (2/5)ma^2$, so we obtain

$$\frac{d\mathbf{v}}{dt} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}} \times \mathbf{v}. \tag{31}$$

The ball therefore undergoes circular motion, with a frequency equal to 2/7 times the frequency of the turntable. This result for the frequency does not depend on initial conditions.

REMARKS: Integrating eq. (31) from the initial time to some later time gives

$$\mathbf{v} - \mathbf{v}_0 = \left(\frac{2}{7}\Omega\right) \hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_0). \tag{32}$$

This may be written in the more suggestive form,

$$\mathbf{v} = \left(\frac{2}{7}\Omega\right) \hat{\mathbf{z}} \times \left(\mathbf{r} - \left(\mathbf{r}_0 + \frac{7}{2\Omega}(\hat{\mathbf{z}} \times \mathbf{v}_0)\right)\right). \tag{33}$$

This equation describes circular motion, with the center located at the point

$$\mathbf{r}_c = \mathbf{r}_0 + (7/2\Omega)(\hat{\mathbf{z}} \times \mathbf{v}_0), \tag{34}$$

and with radius $(7/2\Omega)|\hat{\mathbf{z}} \times \mathbf{v}_0| = 7v_0/2\Omega$. (Eq. (33) does indeed describe circular motion, because it says that $\mathbf{v}$ is always perpendicular to $\mathbf{r} - \mathbf{r}_c$. Hence, the distance to the point $\mathbf{r}_c$ doesn't change.)

There are a few special cases to consider:

- If $v_0 = 0$ (that is, if the spinning motion of the ball exactly cancels the rotational motion of the turntable), then the ball will always remain in the same place (of course).
- If the ball is initially not spinning, and just moving along with the turntable, then $v_0 = \Omega r_0$, so the radius of the circle is $(7/2)r_0$.
- If we want the center of the circle be the center of the turntable, then eq. (34) say that we need $(7/2\Omega)\hat{\mathbf{z}} \times \mathbf{v}_0 = -\mathbf{r}_0$. This implies that $\mathbf{v}_0$ has magnitude $v_0 = (2/7)\Omega r_0$ and points tangentially in the same direction as the turntable moves. (That is, the ball moves at 2/7 times the velocity of the turntable beneath it.)

The fact that the frequency $(2/7)\Omega$ is a rational multiple of Ω means that the ball will eventually return to the same point on the turntable. In the lab frame, the ball will trace out two circles in the time it takes the turntable to undergo seven revolutions. And from the point of view of someone on the turntable, the ball will 'spiral' around five times before returning to the original position.

If we look at a ball with moment of inertia $I = \eta m a^2$ (so a uniform sphere has $\eta = 2/5$), then it is easy to show that the "2/7" in the above result gets replaced by "$\eta/(1+\eta)$". If a ball has most of its mass concentrated at its center (so that $\eta \to 0$), then the frequency of the circular motion goes to 0, and the radius goes to ∞. ■

6. We will divide the solution into the calculations of (1) the frequency of oscillation, (2) the energy loss per oscillation, (3) the amplitude as a function of time.

Frequency of oscillation

In the equilibrium position, the upward force from the spring balances the downward force from gravity on the part of the rope that is in the air. If the rope is moved a distance y (with upward taken to be positive), then the force from the spring changes by $-ky$, while the gravitational force changes by $-(\sigma y)g$. The net force is therefore $F = -(k + \sigma g)y$. This force acts on the rope, which has mass $M = \sigma L$. $F = ma$ therefore gives (the mass of the rope in the air changes slightly, but this effect is negligible when dealing with the "m" in $F = ma$)

$$-(k + \sigma g)y = (\sigma L)\ddot{y}, \tag{35}$$

and so the frequency of oscillation is

$$\omega = \sqrt{\frac{k + \sigma g}{\sigma L}} = \sqrt{\frac{k}{M} + \frac{g}{L}}\,. \tag{36}$$

Remarks: A common incorrect answer for the frequency is $\omega = \sqrt{k/M}$. The g/L term definitely belongs in the correct answer, as can be seen by considering the limit $k \to 0$. (That is, we have a very weak spring which is stretched, say, a kilometer. And a rope of, say, 1 meter hangs from the end.) The spring force doesn't vary much with distance, so it will always pull up with a force of essentially $Mg = (L\sigma)g$. If the rope is moved distance y, then the gravitational force equals $-(L + y)\sigma g$. The net force is therefore $-(\sigma g)y$. $F = ma$ then gives $-(\sigma g)y = (L\sigma)\ddot{y}$. Hence $\omega = \sqrt{g/L}$, which is independent of k (even though the spring force is *not* negligible). The rope will simply bounce up and down, with a frequency determined by its length.

We can be a bit more rigorous in deriving the frequency in eq. (36), by writing down the precise equations of motion for the moving part of the rope. If we let $\ell \equiv L + y$ be the length of rope in the air, then $F = dp/dt$ gives

$$F_{\text{net}} = \frac{d}{dt}\Big((\ell\sigma)\dot{\ell}\Big) = \sigma\ell\ddot{\ell} + \sigma\dot{\ell}^2 = \sigma(L + y)\ddot{y} + \sigma\dot{y}^2 \tag{37}$$

On the way up, the net force on the moving part of the rope is $F_{\text{net}} = -(k + \sigma g)y$, so eq. (37) gives

$$-(k + \sigma g)y = \sigma(L + y)\ddot{y} + \sigma\dot{y}^2. \tag{38}$$

On the way down, the net force on the moving part of the rope is $F_{\text{net}} = -(k + \sigma g)y + F_{\text{floor}}$, where F_{floor} is the force exerted by the floor to bring to rest the atoms that hit the floor. Mass hits the floor at a rate $\sigma|\dot{y}|$, while moving at speed $|\dot{y}|$. The rate of change of momentum of these atoms (i.e., F_{floor}) is therefore $\sigma\dot{y}^2$. Eq. (37) then gives

$$-(k + \sigma g)y = \sigma(L + y)\ddot{y}. \tag{39}$$

To first order in the small quantity y, both eqs. (38) and (39) give the equation of motion in eq. (35), which was derived using the approximate $F = ma$ reasoning, with $m = \sigma L$. ■

Energy loss per oscillation

The position of the rope, relative to the equilibrium position, is essentially equal to

$$x(t) = A(t)\cos(\omega t). \tag{40}$$

The energy loss during the downward motion is fairly straightforward. When a piece of the rope with mass dm hits the floor, it loses a kinetic energy of $(1/2)(dm)v^2$. In a short time dt, we have $dm = |\sigma v\, dt|$. So the loss is $|(1/2)\sigma v^3\, dt|$. From eq. (40), we obtain $v(t) = -\omega A(t)\sin(\omega t)$, so the change in energy during the downward half of the oscillation is

$$\Delta E_{\text{down}} = -\frac{1}{2}\int_0^{\pi/\omega} \sigma\omega^3 A^3 \sin^3(\omega t)\, dt. \tag{41}$$

Letting $\theta \equiv \omega t$, and then using

$$\int_0^\pi \sin^3\theta\, d\theta = \int_0^\pi (1-\cos^2\theta)\sin\theta\, d\theta = \left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\Big|_0^\pi = \frac{4}{3}, \tag{42}$$

gives (using the fact that A is essentially constant throughout an oscillation)

$$\Delta E_{\text{down}} = -\frac{2}{3}\sigma\omega^2 A^3. \tag{43}$$

The energy loss during the upward motion is a little trickier, but the answer turns out to be the same as for the downward motion. When a piece of the rope with mass dm abruptly joins the moving part of the rope, there is an inevitable energy loss. This loss may be calculated as follows. Let a mass dm join the rope at the instant the rope is moving at speed v. Then it gains a kinetic energy of $(1/2)(dm)v^2$. It also gains a momentum of $dP = (dm)v$. The work the spring does in bringing it up to this speed is $W = \int F\, dx = \int Fv\, dt$. The rope is moving at an essentially constant speed v for this short period of time. Hence,

$$W = v\int F\, dt = v(dP) = (dm)v^2. \tag{44}$$

We therefore conclude that half of this work goes into kinetic energy of the mass, and half is lost to heat. The loss to heat is thus $(1/2)(dm)v^2 = (1/2)|\sigma v\, dt|v^2 = |(1/2)\sigma v^3\, dt|$, as in the downward case. The total change in energy per oscillation is therefore

$$\Delta E = \Delta E_{\text{down}} + \Delta E_{\text{up}} = -\frac{4}{3}\sigma\omega^2 A^3. \tag{45}$$

REMARK: The fact that the loss in heat equals the gain in kinetic energy, when an atom joins the moving part of the rope, is a very general result and can easily been understood by looking at things in the frame of the moving rope. In this frame, an atom of mass dm in the heap moves with speed v, and then suddenly comes to rest when it joins the straight part of the rope. The heat loss in the rope's frame (which is the same as the heat loss in the lab frame) is therefore $(dm/2)v^2$, which equals the gain in kinetic energy in the lab frame. The validity of this general result can be traced to the self-evident fact that the speed of the heap with respect to the straight part is the same as the speed of the straight part with respect to the heap. ■

Energy loss per oscillation

The energy of the rope when it has amplitude A is $E = M\omega^2 A^2/2$, thus $dE = M\omega^2 A\, dA$. The number of oscillations in a time dt is $\omega\, dt/2\pi$. Therefore, eq. (45) gives (using $M \approx \sigma L$)

$$\begin{aligned} (\sigma L)\omega^2 A\, dA &= -\left(\frac{\omega\, dt}{2\pi}\right)\left(\frac{4}{3}\sigma\omega^2 A^3\right) \\ \Longrightarrow \qquad \frac{dA}{A^2} &= -\left(\frac{2\omega}{3\pi L}\right) dt. \end{aligned} \tag{46}$$

Integrating this from the start to a time t, and using $A(0) \equiv b$, gives

$$A(t) = \frac{1}{\frac{1}{b} + \frac{2\omega t}{3\pi L}} \,. \tag{47}$$

Remark: For large t, this reduces to

$$A(t) \approx \frac{3\pi L}{2\omega t}\,, \tag{48}$$

which is independent of the initial amplitude b. The $1/t$ behavior implies that the total distance the rope travels barely diverges to infinity, as $t \to \infty$. In terms of $n = \omega t/2\pi$, the number of oscillations undergone, eq. (48) may be written as

$$A(n) \approx \frac{3L}{4n}\,. \qquad \blacksquare \tag{49}$$