Boston Area Undergraduate Physics Competition

SOLUTIONS

1. (a) Imagine the ball sitting on top of the pipe. If it is given a tiny push, it will slide off the pipe and hit the ground with a speed given by $\frac{1}{2}mv^2 = mg(r+h)$. This motion may be reversed. The ball must therefore be thrown with a speed of just greater than $\sqrt{2g(r+h)}$. By conservation of energy, clearly no smaller speed will work.

   (b) *First Solution:* (Based on a solution by Charles Santori and Ron Maimon) Let the parabolic arc of the ball be tangent to the pipe at an angle θ from the top of the pipe. The velocity of the ball there is of the form $(v_\theta \cos\theta, v_\theta \sin\theta)$. The conditions that the parabola reach its maximum over the center of the pipe (any other situation would require more energy) are

$$(v_\theta \cos\theta)t = r\sin\theta, \qquad \text{and} \qquad v_\theta \sin\theta = gt,$$

for some t. These give

$$v_\theta^2 = \frac{gr}{\cos\theta}.$$

Let v be the speed at which the ball is thrown from the ground. Then the energy at the ground is

$$\frac{1}{2}mv^2 = \frac{1}{2}m\Big(\frac{gr}{\cos\theta}\Big) + mg(h + r\cos\theta).$$

Minimizing this function of $\cos\theta$ yields $\cos\theta = 1/\sqrt{2}$, and

$$v^2 = g(2\sqrt{2}r + 2h).$$

Note that $\theta = 45^{\circ}$, independent of the ratio of h to r .

Second Solution: Let the parabolic arc of the ball have a maximum height l and span a distance $2d$ on the ground. If the initial velocity of the ball is (v_x, v_y) and it is in the air for time $2t$, then

$$v_x t = d, \qquad v_y = gt, \qquad \text{and} \qquad \frac{1}{2}gt^2 = l.$$

Eliminating t gives

$$v^2 = g(\frac{d^2}{2l} + 2l).$$

We want to minimize this, given that the parabola goes over the pipe. The parabola is given by

$$y = -\frac{l}{d^2}x^2 + l.$$

The pipe's cross section is given by

$$(y-h)^2 + x^2 = r^2.$$

Solving for the y value of their intersection gives

$$y = \frac{1}{2l}(2lh + d^2 \pm \sqrt{d^4 + d^2(4lh - 4l^2) + 4l^2r^2}).$$

We want the parabola to be tangent to the pipe, i.e., the discriminant here is zero. Therefore, $d^2 = 2l(l - h \pm \sqrt{(l-h)^2 - r^2})$. The minus sign is the physically relevant one (the plus sign corresponds to negative values for x^2). The above expression for v^2 now yields

$$v^2 = g(3l - h - \sqrt{(l-h)^2 - r^2}).$$

Minimizing this function of l yields $l = h + \frac{3}{2\sqrt{2}}r$, and

$$v^2 = g(2\sqrt{2}r + 2h).$$

[Note that this is indeed less than the value at $l = h + r$, namely $v^2 = g(3r + 2h)$, a common, incorrect answer to this problem. Touching the pipe tangentially at the top is a limiting case of having two tangent points very close together. A possible candidate for an optimal path not covered in the above solution is one which is tangent to the top of the pipe, and which has a larger value for d than the one just mentioned. But we know that v^2 increases with d, for constant l.]

2. (a) If the partition is removed very quickly, no work is done on the gas. Therefore, the temperature remains equal to T.

   (b) Due to the cylindrical symmetry, the component of B in the tangential direction around the wire is, from Ampere's Law, $\frac{\mu_o I}{2\pi r}$. ¿From a few applications of the right hand rule, it is easy to see that the currents in the plane will give only tangential components.

   (c) Energy is conserved, but the angular momentum of the puck is not. The force from the string is not a central force; or rather, the "center" keeps changing. (Angular momentum *is* conserved, of course, if the earth is included.)

3. It is given that $L \gg r$. Consider a sphere of radius aL $(L \gg aL \gg r)$ surrounding one of the metal spheres. Since $L \gg aL$, the current flow is approximately spherically symmetric out to aL. The resistance out to aL is computed by considering many spherical shells of thickness dr in series. The resistance of a shell is $\rho dr/(4\pi r^2)$. Integrating this gives a resistance of $\frac{\rho}{4\pi}(\frac{1}{r} - \frac{1}{aL})$ out to a radius aL. The other sphere has the same resistance out to a radius aL. So the total resistance between the two metal spheres is $\frac{\rho}{2\pi}(\frac{1}{r} - \frac{1}{aL})$ plus the resistance between the spheres of radius aL. This latter resistance is less than that of a cylinder of radius aL and length L, which has a resistance of $\rho L/(\pi a^2 L^2)$. This is negligible compared to the $1/r$ term

above, as long as $a^2L \gg r$. If L is large enough compared to r, then it is possible to pick an a so that $L \gg aL \gg a^2L \gg r$. Therefore, keeping only the leading term,

$$R = \frac{\rho}{2\pi r}.$$

4. (a) We will use complex impedances to solve this problem, and use the equation analogous to Ohm's law: $I = \mathrm{Re}(V_o e^{i\omega t}/Z)$, where Z is the complex impedance. Since we have a semi-infinite circuit here, adding another "box" to the circuit shouldn't change the impedance. Therefore,

$$Z = i\omega L + \frac{Z\frac{1}{i\omega C}}{Z + \frac{1}{i\omega C}}.$$

Solving for Z gives

$$Z = \frac{1}{2}\left(i\omega L \pm L\sqrt{\frac{4}{LC} - \omega^2}\right).$$

There are two cases to consider:

i. $\omega > 2/\sqrt{LC}$:
The impedance is purely imaginary,

$$Z = \frac{i}{2}\left(\omega L + L\sqrt{\omega^2 - \frac{4}{LC}}\right).$$

The plus sign is selected because for $C \to \infty$, Z must become $i\omega L$ (alternatively, for $\omega \to \infty$, Z must go to ∞). $I = \mathrm{Re}(V_o e^{i\omega t}/Z)$ now gives

$$I(t) = \frac{2V_o}{\omega L + L\sqrt{\omega^2 - \frac{4}{LC}}} \sin \omega t.$$

ii. $\omega < 2/\sqrt{LC}$:
The impedance is

$$Z = \frac{1}{2}\left(i\omega L + L\sqrt{\frac{4}{LC} - \omega^2}\right).$$

The plus sign is chosen, since the real part of an impedance is positive. $I = \mathrm{Re}(V_o e^{i\omega t}/Z)$ now gives

$$I(t) = \frac{V_o}{\sqrt{L/C}} \cos(\omega t - \phi),$$

where $\tan \phi = \omega / \sqrt{\frac{4}{LC} - \omega^2}$.

(b) i. $\omega > 2/\sqrt{LC}$:

¿From $V(t) = V_o \cos\omega t$ and the expression for I above, we see that the average power delivered by the source goes like the average of $\sin\omega t\cos\omega t$, so

$$\langle P\rangle = 0$$

(the real part of the impedance is zero).

ii. $\omega < 2/\sqrt{LC}$:

The average of $P = IV$ is the average of $V_o \cos\omega t$ times the expression for I given above, so we obtain

$$\langle P\rangle = \frac{1}{4}CV_o^2\sqrt{\frac{4}{LC} - \omega^2}.$$

This is non-zero because the impedance has a real part. The power delivered by the source would be zero for a *finite* circuit. In our infinite case, the energy of the source is dissipated in the form of a wave propagating along the circuit.

5. (a) This will be a little wordy since we're going to try to do this without drawing any pictures. Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, a sector of a circle, S. (You may want to stop reading at this point, and try to solve it yourself.)

If the cone is very sharp, then S will look like a thin "pie piece". If the cone is very wide, with a shallow slope, then S will look like a pie with a piece taken out of it. Points on the straight-line boundaries of S are identified. Let P be the location of the lasso's knot. Then P appears on each straight-line boundary, at equal distances from the tip of S. Let β be the angle of the sector S.

The path of the lasso's loop must be a straight line on S. (The rope will take the shortest distance between two points, since there is no friction, and rolling the cone onto a plane does not change distances.) Such a straight line between the two identified points P is possible only if the sector S is smaller than a semicircle, i.e., $\beta < 180^o$.

Let C denote a cross sectional circle a distance d from the top of the conical mountain, and let R equal the ratio of the circumference of C to d. Then a semicircle S implies that $R = \pi$. This then implies that the radius of C is equal to $d/2$. Therefore, $\alpha/2 = \sin^{-1}(1/2)$. So we find that if the climber is to be able to climb up along the mountain, then

$$\alpha < 60^o.$$

Having $\alpha < 60^o$ guarantees that there is a loop around the cone of shorter length than the distance straight to the peak and back.

[When viewed from the side, the rope should appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies $\alpha < 90^{\circ}$. This is not the case because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop; but the loop must certainly have a discontinuous change in slope where the knot is. (For planar, triangular mountains, the answer is $\alpha < 90^{\circ}$.)]

(b) Same strategy. Roll the cone onto a plane. If the mountain very steep, the climber's position can fall by means of the loop growing larger; if the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in the position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the figure S on a plane, the condition is that if we move P a distance l up (down) along the mountain, the distance between the identified points P decreases (increases) by l. We must therefore have $2\sin(\beta/2) = 1$. So $\beta = 60^{\circ}$, which corresponds to

$$\alpha = 2\sin^{-1}(1/6).$$

There is exactly one angle for which the climber can climb up along the mountain.

Another way to see that $\beta = 60^{\circ}$ is to note that the three directions of rope emanating from the knot must all have the same tension, since the deluxe lasso is one continuous piece of rope. Therefore they must have 120° angles between themselves. This implies that $\beta = 60^{\circ}$.

(c) Roll the cone N times onto a plane. The resulting figure S_N is a sector of a circle divided into N equal sectors, each representing a copy of the cone. S_N must be smaller than a semicircle, so we must have $R < \pi/N$. Therefore,

$$\alpha < 2\sin^{-1}(\frac{1}{2N}).$$

(d) Roll the cone N times onto a plane. From the above reasoning, we want $N\beta = 60^{\circ}$. Therefore,

$$\alpha = 2\sin^{-1}(\frac{1}{6N}).$$

6. (a) The main point of this problem is that when the pivot point of the pencil changes (i.e., when a new spoke hits the plane), the speed of the axis changes suddenly, and kinetic energy is lost. Only the velocity component perpendicular to the new spoke survives from the previous velocity (which was perpendicular to the old spoke). The loss in kinetic energy is proportional to the square of the velocity right before the change. When the speed has increased to a magnitude where this loss in kinetic energy equals the gain from the change in potential energy, the pencil will not go any faster.

(b) We may as well do the problem for a general number of spokes, N, and then let $N = 6$. Let α be the angle of inclination of the plane, v_o be the speed of the axis right before a new spoke hits, and $\beta = 2\pi/N$. Then the speed of the axis right after the new spoke hits is $v_o \cos\beta$.

Equating the change in potential energy during an Nth of a rotation and the kinetic energy loss due to the changing of the contact spoke gives $\frac{1}{2}mv_o^2(1-\cos^2\beta) = mgr(2\sin\frac{\beta}{2})\sin\alpha$. So in the "steady" state, the maximum speed v_o of the axis is given by

$$v_o^2 = \frac{4gr\sin\frac{\beta}{2}\sin\alpha}{\sin^2\beta}.$$

For $N = 6$ and $\beta = \pi/3$, we have

$$v_o^2 = \frac{8}{3}gr\sin\alpha.$$

If conditions have been set up (assuming contact is maintained with the plane, as stated in the problem) such that a non-zero v_o exists, it must be this.

(c) If $\alpha < \beta/2$, then right after the pivot point changes, the axis must actually move upward before falling down along the plane. For a non-zero v_o to exist, we must ensure that the axis is moving fast enough to get over this "bump" (remember that an initial kick to the pencil is allowed). So (assuming $\alpha < \beta/2$) the height the axis must climb is $r(1-\cos(\frac{\beta}{2}-\alpha))$. The speed at which the axis starts this climb is $v_o\cos\beta$. Therefore, we must have $\frac{1}{2}m(v_o\cos\beta)^2 > mgr(1-\cos(\frac{\beta}{2}-\alpha))$. Using the expression for v_o above,

$$\frac{2\sin\frac{\beta}{2}\sin\alpha\cos^2\beta}{\sin^2\beta} > (1-\cos(\frac{\beta}{2}-\alpha)).$$

For $N = 6$ and $\beta = \pi/3$, we have

$$\frac{\sqrt{3}}{2}\cos\alpha > 1 - \frac{5}{6}\sin\alpha.$$

Squaring and solving for $\sin\alpha$ gives

$$\sin\alpha > \frac{15-6\sqrt{3}}{26}.$$

(One may estimate this using $\sin\alpha \approx \alpha$ to obtain an angle of about 10^o.)

(d) The axis of the pencil moves on a circular arc around the pivot point. The force of gravity along the contact spoke must account for the centripetal acceleration of the axis.

The maximal centripetal acceleration occurs right before the pivot point changes, and is equal to mv_o^2/r. The minimal force along the spoke from gravity also occurs right before the pivot point changes, and is $mg\cos(\alpha + \frac{\beta}{2})$. Using the expression for v_o above, $mv_o^2/r \leq mg\cos(\alpha + \frac{\beta}{2})$ becomes

$$\tan\alpha \leq \frac{\sin^2\beta}{4+\sin^2\beta}\cot(\beta/2).$$

For $N = 6$ and $\beta = \pi/3$, this gives

$$\tan\alpha \leq \frac{3\sqrt{3}}{19}.$$

(A small angle approximation shows this to be about 15°.)

(e) For small β and α we find, using the expressions in (b), (c), and (d):

The expression for v_o becomes

$$v_o^2 = 2gr\frac{\alpha}{\beta}.$$

The condition to make it over the "bump" becomes

$$\frac{\alpha}{\beta} > \frac{1}{2}(\alpha - \frac{\beta}{2})^2.$$

For small α and β this implies $\alpha > 0$ (up to third order corrections).

The condition to stay on the plane becomes

$$\alpha \leq \beta/2.$$

In other words, if the angle of inclination is increased until the pencil starts to roll on its own, then it will eventually leave the plane.

(f) Combining the large-N answers to (b) and (d) gives

$$v_o \leq \sqrt{gr},$$

which is independent of N.

This last result can be obtained in a simpler way, which makes it less surprising. An inverted pendulum's centripetal acceleration mv^2/r must be accounted for by the weight mg on the spoke. Therefore $v^2 \leq gr$. The tilt of the plane will change this by factors essentially equal to 1, for small α.