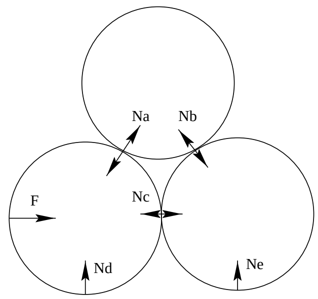
1. Apart from gravity, the forces on the cylinders include the applied force of magnitude F, normal forces between the bottom two cylinders and the ground with magnitudes N_d and N_e, and three pairs of forces with magnitudes N_a, N_b and N_c between the cylinders (as shown on the diagram).



- The low limit on the acceleration is given by the condidtion that the two bottom cylinders lose contact, i.e., $N_c = 0$. Then, the horizontal component of the force on the right cylinder is

$$Ma = N_b \sin(\pi/6) = N_b/2 \tag{1}$$

The vertical components of the forces on the top cylinder:

$$Mg = N_b \cos(\pi/6) + N_a \cos(\pi/6) \tag{2}$$

and the left cylinder:

$$N_a \sin(\pi/6) = N_a/2 = F - Ma = 2Ma, \tag{3}$$

Eliminating N_a and N_b

$$Mg = (2Ma + 4Ma)\sqrt{3}/2 \tag{4}$$

and $a_{min} = \frac{g}{3\sqrt{3}}$

- When the acceleration is increased beyond a certain value, the top cylinder will lose contact with the right cylinder. It corresponds to vanishing N_b. In this case for the top cylinder the vertical components of the forces:

$$Mg = N_a \cos(\pi/6) \tag{5}$$

For horizontal components we get

$$Ma = N_a \sin(\pi/6) = N_a/2, \tag{6}$$

and $a_{max} = \frac{1}{2}2g/\sqrt{3} = g/\sqrt{3} = 3a_{min}$

2. The force on the particle in the magnetic field is $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \vec{B}$. In cylindrical coordinates, where $\mathbf{r} = r\hat{\mathbf{r}}$, $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$, and $\ddot{\mathbf{r}} = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{\mathbf{r}}$, at any given time the tangential projection of the force on the particle is

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = q\dot{r}B \tag{7}$$

By multiplying both sides of the equation by r, we get

$$2mr\dot{r}\dot{\theta} + r^2\ddot{\theta} = qr\dot{r}B, \tag{8}$$

or, extracting the full time differential on the left,

$$\frac{md(r^2\dot{\theta})}{dt} = \frac{qB(r)rdr}{dt}. \tag{9}$$

which could also be seen as the rate of change of angular momentum of the particle $L = mr^2\dot{\theta}$ being equal to the torque about the center of the circular region. Integrating (9) over the time it takes for the particle to exit the circular region,

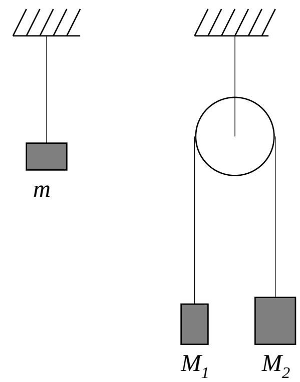
$$r^2\dot{\theta}|_{t,r=0}^{t=T,r=R} = \int_{r=0}^{r=R} qBrdr \tag{10}$$

The right side is $q/2\pi$ times the total flux through the circular region, which is zero, so $\dot{\theta}$ at the time the particle leaves the region must be zero. Since the total velocity of the particle is non-zero (a static magnetic field does not do work), the velocity of the particle then must point radially outward.

3. *First Solution:*

Consider the following auxiliary

Problem: Two set-ups are shown in the figure.



(The first contains a hanging mass m. The second contains a hanging pulley, over which two masses, M_1 and M_2, hang.) Both supports have acceleration a_s downward. What should m be, in terms of M_1 and M_2, so that the tension in the top string is the same in both cases?

Answer: In the first case, we have

$$mg - T = ma_s. \tag{11}$$

In the second case, let a be the acceleration of M_2 relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} M_1 g - \frac{T}{2} &= M_1(a_s - a), \\ M_2 g - \frac{T}{2} &= M_2(a_s + a). \end{aligned} \tag{12}$$

Note that if we define $g' \equiv g - a_s$, then we may write these three equations as

$$\begin{aligned} mg' &= T, \\ M_1 g' &= \frac{T}{2} - M_1 a, \\ M_2 g' &= \frac{T}{2} + M_2 a. \end{aligned} \tag{13}$$

The last two give $4M_1M_2g' = (M_1 + M_2)T$. The first equation then gives

$$m = \frac{4M_1M_2}{M_1 + M_2}. \tag{14}$$

Note that the value of a_s is irrelevant. (We effectively have a fixed support in a world where the acceleration from gravity is g'.) This problem shows that the two-mass system in the second case may be equivalently treated as a mass m, as far as the upper string is concerned. ∎

Now let's look at our infinite Atwood machine. Start at the bottom. (Assume the system has N pulleys, where $N \to \infty$.) Let the bottom mass be x. Then the above problem shows that the bottom two masses, M and x, may be treated as an effective mass $f(x)$, where

$$f(x) = \frac{4x}{1 + (x/M)}. \tag{15}$$

We may then treat the combination of the mass $f(x)$ and the next M as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass M and a mass $f^{(N-1)}(x)$ hanging over the top pulley.

We must determine the behavior of $f^N(x)$, as $N \to \infty$. The behavior is obvious by looking at a plot of $f(x)$ (which we'll let the reader draw). (Note that $x = 3M$ is a fixed point of f, i.e., $f(3M) = 3M$.) It is clear that no matter what x we start with, the iterations approach $3M$ (unless, of course, $x = 0$). So our infinite Atwood machine is equivalent to (as far as the top mass is concerned) just the two masses M and $3M$.

We then easily find that the acceleration of the top mass is (net downward force)/(total mass) $= 2Mg/(4M) = g/2$.

NOTE: As far as the support is concerned, the whole apparatus is equivalent to a mass $3M$. So $3Mg$ is the weight the support holds up.

Second Solution:

Note that if the gravity in the world were multiplied by a factor η, then the tension in all the strings would likewise be multiplied by η. (The only way to make a tension, i.e., a force, is to multiply a mass times g.) Conversely, if we put the apparatus on another planet and discover that all the tensions are multiplied by η, then we know the gravity there must be ηg.

Let the tension in the string above the first pulley be T. Then the tension in the string above the second pulley is $T/2$ (since the pulleys are massless). Let the acceleration of the second pulley be a_{p2}. Then the second pulley effectively lives in a world where the gravity is $g - a_{p2}$. If we imagine holding the string above the second pulley and accelerating downward at a_{p2} (so that our hand is at the origin of the new world), then we really haven't changed anything, so the tension in this string in the new world is still $T/2$.

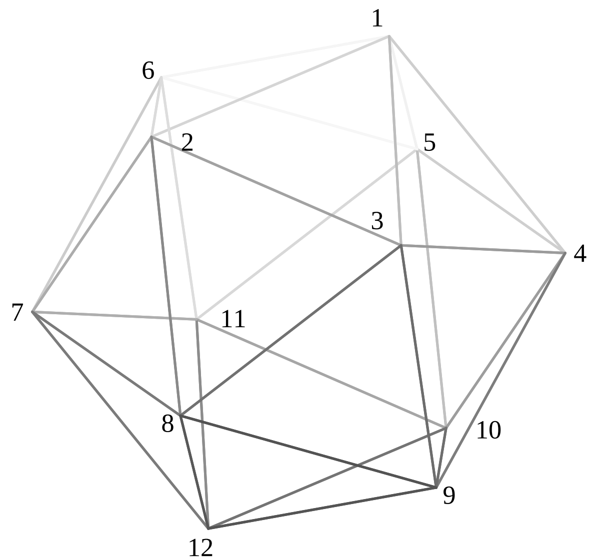
But in this infinite setup, the system of all the pulleys except the top one is the same as the original system of all the pulleys. Therefore, by the arguments in the first paragraph, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_{p2}}. \tag{16}$$

Hence, $a_{p2} = g/2$. (Likewise, the relative acceleration of the second and third pulleys is $g/4$, etc.) But a_{p2} is also the acceleration of the top mass. So our answer is $g/2$.

Note that $T = 0$ also makes eq. (16) true. But this corresponds to putting a mass of zero at the end of a finite pulley system.

4. This problem is just another variation of the popular resistor cube problem, offered, among other places, in BAUPC'96.



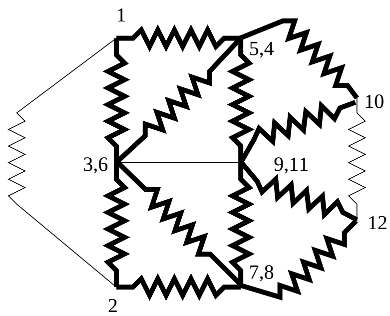
There are two solutions; the second is by far more general and will work with any symmetric polyhedron (or a resistor graph).

(a) A straightforward solution

The trick is to simplify the diagram by connecting with a wire points which have equal potential anyway. Let's assume that the leads are connected to vertices 1 and 2. Then due to symmetry consideration the following sets of points will have equal potentials:

- $3, 6, 9, 11$
- $4, 5$
- $7, 8$

The diagram reduces to:



Thicker pen used for resistances of 1/2 Ohm.

Further simplification is achieved by shorting the middle points of $1-2$ and $10-12$ resistors to the equipotential line $3,6-9,11$. Now the resistance between 1 and 2 can be calculated as (right to left, in Ohm)

$$R_{1-2}/2 = (((1/2||1/2) + 1/2)||1/2||1/2) + 1/2)||1/2||1/2 = 11/60, \tag{17}$$

where $a||b = ab/(a+b)$

Resistance between the points 1 and 2 is 11/30 Ohm.

(b) A more elegant solution

Consider the following current configuration. Current I is being fed into vortex 1. The other 11 vortices are connected to an external circuit in such a way that each of them drains the same current, *i.e.*, $I/(12-1)$. Symmetry arguments tell us that all of the 5 resistors (edges), emanating at vortex 1 will be carrying the same current, namely $I/5$. The voltage across the resistor $1-2$ will then be $I/5 \times 1Ohm$.

Now let's consider a second configuration. This time, vortex 2 is *drawing* current I, and each of the other vortices $(1, 3-12)$ is connected to an external circuit in such a way that the current flowing into each of them is $I/(12-1)$

The next step is to superimpose the two configurations. It is possible to do because the resistors are linear, *i.e.*, $V = const \times I$. For each resistor, the total current flowing is the sum of the currents in the two configurations. Same holds for the voltage drops. The currents flowing to the outside circuits from vertices $3-12$ are zero, and so the connections to those circuits can be severed.

The current flowing through vertices 1 and 2 are equal in magnitude and are $I + I/(12-1)$. The voltage drop across resistor $1-2$ is $V = 2 \times I/5 \times 1Ohm$. The resistance measured between points 1 and 2 is then

$$R = \frac{2I/5}{12/11I} Ohm = 11/30 Ohm \tag{18}$$

Note that the numerator is equal to the total number of vortices minus one, and the denominator is equal to the total number of resistors (edges), the latter being equal to the

$$\frac{1}{2}\, number\ of vortices \times number\ of\ resistors\ eminating\ from\ each\ vortex \tag{19}$$

We leave it to the reader to verify the formula for other symmetric polyhedra.

5. Let the angular speeds of the tops be ω_i, starting with the top one (so $\omega_1 \equiv \omega$). Let I be the moment of inertia of each top around its symmetry axis. Let Ω be the angular speed of precession.

We will use $\vec{\tau} = d\vec{L}/dt$ on each top. We therefore must determine $d\vec{L}/dt$ and the torque $\vec{\tau}$ for each top.

- $d\vec{L}/dt$:

  If the ω_i's are large enough (as we are assuming), then the angular momentum of the ith top will have magnitude essentially equal to $L_i = I\omega_i$, and $\vec{L}_i$ will point along the symmetry axis. (In other words, we can neglect the angular momentum due to the slow angular velocity of precession. We will see below that $\Omega \propto 1/\omega$.)

  The tip of $\vec{L}_i$ will trace out a circle of radius $L_i \sin\theta$, with angular speed Ω. Therefore,

  $$\left|\frac{d\vec{L}_i}{dt}\right| = \Omega L_i \sin\theta = \Omega I \omega_i \sin\theta, \tag{20}$$

  and $d\vec{L}_i/dt$ points tangentially around the circle.

- $\vec{\tau}$:

  None of the N tops are accelerating in the vertical direction. Therefore, the forces on the bottom top are NMg upward (to balance the weight of all the tops) at its lower end, and $(N-1)Mg$ downward (to keep up the other $N-1$ tops) at its upper end. The torque on the bottom top (around its CM) therefore has magnitude $(2N-1)Mgr\sin\theta$, where r is half the length of a top. It points perpendicular to the page.

  It is easy to see that the torque on the second-to-bottom top has magnitude $(2N-3)Mgr\sin\theta$, and so on, until the torque on the top top is $Mgr\sin\theta$.

  So the torque on the ith top has magnitude $(2i-1)Mgr\sin\theta$.

Equating $\vec{\tau}$ with $d\vec{L}_i/dt$ gives

$$(2i-1)Mgr\sin\theta = \Omega I \omega_i \sin\theta. \tag{21}$$

Therefore,

$$\omega_i = (2i-1)\omega_1 \equiv (2i-1)\omega. \tag{22}$$

NOTE: As a double-check, the reader can verify that these ω_i's make $\vec{\tau} = d\vec{L}/dt$ true, where $\vec{\tau}$ and $\vec{L}$ are the total torque and angular momentum about the CM of the entire system.

6. (a) Let r be the distance from A to the pulley, and let θ be the angle of the string to A (w.r.t. to the vertical). Let T be the tension in the string (T will depend on r and θ). Then $F = ma$, along the direction of the string, on masses A and B gives, respectively,

$$\begin{aligned} T - mg\cos\theta &= mr\dot{\theta}^2 - m\ddot{r}, \\ T - mg &= m\ddot{r}. \end{aligned} \tag{23}$$

These combine to give $2\ddot{r} = r\dot{\theta}^2 - g(1-\cos\theta)$. Using the small angle approximation for $\cos\theta$, we have

$$2\ddot{r} = r\dot{\theta}^2 - \frac{1}{2}g\theta^2. \tag{24}$$

Consider two cases of the motion.

- *Immediately after A is given the kick:*
At this time, r is essentially not changing. Hence, the motion is approximately that of a pendulum of length ℓ. Therefore, θ and $\dot{\theta}$ take the form

$$\theta \approx \frac{\epsilon}{r}\sin\omega t, \qquad \text{and} \qquad \dot{\theta} \approx \frac{\omega\epsilon}{r}\cos\omega t, \tag{25}$$

where $\omega = \sqrt{g/r}$, and $r = \ell$. Plugging these expressions into eq. (24), and using $\omega^2 = g/r$, gives

$$2\ddot{r} = \frac{g\epsilon^2}{r^2}\cos^2\omega t - \frac{g\epsilon^2}{2r^2}\sin^2\omega t. \tag{26}$$

The $\sin^2\omega t$ and $\cos^2\omega t$ terms average to $1/2$ over a few periods. Therefore, the average value of $\ddot{r}$, over a few periods, is

$$\ddot{r} = \frac{\epsilon^2 g}{8r^2}. \tag{27}$$

This is positive. Hence, mass B initially starts to climb.

- *After B has risen a significant distance:*
In this case, r will be changing, so the motion won't look exactly like that of a pendulum.[1] However, it turns out that $\ddot{r}$ is still given by eq. (27), (except that both ϵ and r will have changed; we will show how ϵ changes, in part (b)). Let us see why this is true.

[1]We will find in part (b) that after a while, $\dot{r}$ is roughly the same size as the speed of the pendulum when it passes through $\theta = 0$. So the motion is not like a that of a pendulum.

Eq. (24) is still valid when $\dot{r} \neq 0$. The quantities that require some care are the θ and $\dot{\theta}$ in eq. (25). It is not obvious that these expressions are valid, since the motion doesn't resemble that of a pendulum. However, these forms of θ and $\dot{\theta}$ *are* still true, for the following reason.

At a given time, let $\dot{r} = v$. Consider a frame moving at downward at constant speed v. In this frame, the motion of A looks like that of a pendulum. The acceleration due to gravity in this frame is still g. And most importantly, the fractional change in r, over one period, is very small (because $\dot{r}$ is very small, as we shall see). Hence, the motion looks like that of a pendulum with definite frequency $\omega = \sqrt{g/r}$. And since the frame moves at constant speed, $\ddot{r}$ in this frame equals $\ddot{r}$ is the lab frame. So eq. (27) is still valid. Hence, $\ddot{r}$ is always positive, and so r increases. Therefore, mass B is the one that hits its pulley.

(b) From eq. (27), the initial acceleration of B is $a_i = \epsilon^2 g/8\ell^2$. If this were the acceleration at all times, then the speed of B when it hits the pulley would be $\sqrt{2a_i\ell} = \sqrt{\epsilon^2 g/4\ell}$. This, however, is not correct, because as time goes by both ϵ and ℓ in eq. (27) change, thereby making $\ddot{r}$ change as B rises. To determine how $\ddot{r}$ behaves, it will suffice to determine how ϵ depends on r.

Claim: *The amplitude ϵ scales with r according to $\epsilon \propto r^{1/4}$.*

Proof: We will find ϵ as a function of r by looking at the kinetic energy of A at $\theta = 0$.

The kinetic energy of A, at $\theta = 0$, decreases in time. This is most easily seen by noting that B picks up KE, since it begins to move upward; therefore A must lose KE. (Consider an instant when $\theta = 0$. The total potential energy of the system is the same as when it started. Therefore, the KE gained by B equals the KE lost by A.)

This relationship between the KE's of A and B, at $\theta = 0$, may be expressed as

$$\frac{d}{dr}\left(\frac{1}{2}mr^2\dot{\theta}^2_{\theta=0} + \frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dr}\left(\frac{1}{2}m\dot{r}^2\right). \tag{28}$$

Now, $d\left(\frac{1}{2}m\dot{r}^2\right)$ is the work done on B, which is $dW_B = (T - mg)dr$. From eqs. (23) and (27), we find

$$\frac{dW_B}{dr} = T - mg = \frac{mg\epsilon^2}{8r^2}, \tag{29}$$

where we have taken an average over a few periods to obtain the second equality.

Also, eq. (25) gives $\dot{\theta}^2_{\theta=0} = (\epsilon\omega/r)^2$, with $\omega = \sqrt{g/r}$. So eq. (28) becomes

$$\frac{d}{dr}\left(\frac{mg\epsilon^2}{2r}\right) = -2\frac{dW_B}{dr} = -\frac{mg\epsilon^2}{4r^2}. \tag{30}$$

Taking the derivative and simplifying yields

$$\frac{1}{\epsilon}\frac{d\epsilon}{dr} = \frac{r}{4}. \tag{31}$$

Integrating and then exponentiating gives $\epsilon_r \equiv \epsilon(r) = Cr^{1/4}$. We therefore find that under this very slow change in r, the amplitude ϵ scales like $r^{1/4}$. ■
The initial condition $\epsilon_\ell \equiv \epsilon$, gives

$$\epsilon_r = \epsilon\left(\frac{r}{\ell}\right)^{1/4}. \tag{32}$$

The acceleration in eq. (27) then becomes

$$\ddot{r} = \left(\frac{\epsilon^2 g}{8\sqrt{\ell}}\right)\frac{1}{r^{3/2}}. \tag{33}$$

Multiplying by $\dot{r}$ and integrating gives

$$\frac{\dot{r}^2}{2} = \frac{\epsilon^2 g}{4\ell} - \left(\frac{\epsilon^2 g}{4\sqrt{\ell}}\right)\frac{1}{\sqrt{r}}, \tag{34}$$

where the constant of integration on the right-hand side was determined using the condition that $\dot{r} = 0$ when $r = \ell$.
Finally, plugging in $r = 2\ell$ gives

$$\dot{r}_{r=2\ell} = \sqrt{\frac{\epsilon^2 g}{2\ell}\left(1 - \frac{1}{\sqrt{2}}\right)}. \tag{35}$$

NOTE: This $\dot{r}$ is the same order of magnitude as the angular speed of the 'pendulum' at $\theta = 0$, namely $r\dot{\theta} = \epsilon_r\omega_r$, because this is (up to factors of order 1) equal to $\epsilon_\ell\omega_\ell = \epsilon\sqrt{g/\ell}$.