The Boston Area Undergraduate Physics Competition

April 20, 1996, 1:00-4:00 pm
Jefferson 250
Harvard University

Each of the six questions is worth 10 points.

1. (a) A ball, B_2, with (very small) mass m_2 sits on top of another ball, B_1, with (very large) mass m_1 [see the figure below]. The bottom of B_1 is at a height h above the ground, and the bottom of B_2 is at a height $h + d$ above the ground. The balls are dropped. How high does the top ball bounce?

   (You may work in the approximation where m_1 is much heavier than m_2. Assume that the balls bounce elastically. And assume, for the sake of having a nice clean problem, that the balls are initially separated by a small distance, and that the balls bounce instantaneously.)

   (b) n balls, $B_1, \ldots, B_n$, with masses $m_1, m_2, \ldots, m_n$, respectively ($m_1 \gg m_2 \gg \cdots \gg m_n$), sit in a vertical stack [see the figure below]. The bottom of B_1 is at a height h above the ground, and the bottom of B_n is at a height $h + \ell$ above the ground. The balls are dropped. In terms of n, how high does the top ball bounce?

   (Work in the approximation where m_1 is much heavier than m_2, which is much heavier than m_3, etc., and assume that the balls bounce elastically. Also, make the "nice clean problem" assumptions as in part (a).)

   If $h = 1$ meter, what is the minimum number of balls needed in order for the top one to bounce to a height of at least 1 kilometer?

   (Assume the balls still bounce elastically [which is in reality not likely]. Ignore wind resistance, etc., and assume that l is negligible here.)

2. Each of the two unrelated parts of this question is worth 5 points:

   (a) Let $\vec{E}$ be the electric field due to a circular ring of radius R and uniform charge density. Show that somewhere along the surface of a cylinder of radius $R/2$, whose axis passes through the center of the ring and is perpendicular to the plane of the ring [see the figure below], $\vec{E}$ is parallel to the axis of the cylinder.

   (b) Two cylindrical containers, A and B, have the same shape and contain equal volumes of water. In addition to the water, B contains an immersed balloon, attached to the bottom with a string [see the figure below]. (Assume that the mass of the air in the balloon is negligible.) The following reasoning claims that the pressure in the water at the bottom of A is the equal to the pressure in the water at the bottom of B. Is this reasoning correct or incorrect? (Explain why)

   Reasoning: The total upward force exerted by the bottom of container A is equal to the weight of the water in A; likewise for B. But the force exerted by the bottom of A is equal to the pressure times the area of the bottom; likewise for B. Therefore, since the areas of the bottoms are the same, and the weights of the water are the same, the pressures at the bottoms must be the same.

3. In the figure below, the vertices $ABCDEFGH$ form a cube. Each of the twelve edges and each of the twelve diagonals on the surface (two diagonals on each of the six faces) is a 1Ω resistor. (In other words, if ℓ is the length of an edge, then every segment of length ℓ or $\sqrt{2}\,\ell$ is a 1Ω resistor. We have drawn a few of the diagonal resistors in the figure, but we have not drawn them all, for the sake of having a readable figure.)

   Find the resistance between points A and C.

4. (a) Two circular rings, in contact with each other, stand in a vertical plane [see the figure below]. Each has radius R. A small ball, with mass m and negligible size, bounces elastically back and forth between the rings. (Assume that the rings are held in place, so that they always remain in contact with each other.) Assume that initial conditions have been set up so that the ball's motion forever lies in one parabola. Let this parabola hit the rings at an angle θ from the horizontal.

   i. Let $\Delta P_x(\theta)$ be the magnitude of the change in the horizontal component of the ball's momentum, at each bounce. For what angle θ is $\Delta P_x(\theta)$ maximum?

   ii. Let S be the speed of the ball just before or after a bounce. And let $\overline{F}_x(\theta)$ be the average (over a long period of time) of the magnitude of the horizontal force needed to keep the rings in contact with each other (for example, the average tension in a rope holding the rings together). Consider the two limits: (1) $\theta \approx \epsilon$, and (2) $\theta \approx \pi/2 - \epsilon$, where ϵ is very small.

      A. Derive approximate formulas for S, in these two limits.

      B. Derive approximate formulas for $\overline{F}_x(\theta)$, in these two limits.

      (You may use the small-angle approximations $\sin \epsilon \approx \epsilon$, $\cos \epsilon \approx 1 - \epsilon^2/2$, etc. Give your answers to leading order in ϵ.)

(b) Consider the more general case where a ball bounces back and forth between a surface defined by $f(x)$ (for $x > 0$) and $f(-x)$ (for $x < 0$) [see the figure below]. Again, assume that initial conditions have been set up so that the ball's motion forever lies in one parabola (the ball bounces back and forth between the contact points at $(x_0, f(x_0))$ and $(-x_0, f(x_0))$, for some x_0).

   i. Let $\Delta P_x(x_0)$ be the absolute value of the change in the horizontal component of the ball's momentum, at each bounce. For what function $f(x)$ is $\Delta P_x(x_0)$ independent of the contact position x_0?

   ii. Let $\overline{F}_x(x_0)$ be the average of the magnitude of the horizontal force needed to keep the two halves of the surface together. For what function $f(x)$ is $\overline{F}_x(x_0)$ independent of the contact position x_0?

5. Consider the following rigid structure: A mass M is located at the vertex of an angle formed by two sticks [see the figure below], each having length ℓ and negligible mass. The fixed angle between the sticks is θ.

   In this problem you are asked to determine, for small θ, approximately how long this structure can rock back and forth.

   More concretely: hold the structure so that one of the sticks (say, the left one) is vertical. Then give the mass an infinitesimal push, so that eventually the right stick will hit the ground. Let this occur at time t_0. (Assume that there is sufficient friction between the sticks and the ground so that the sticks do not slide, and assume that the sticks do not bounce when they hit the ground.) The left stick will lose contact with the ground, and the system will then pivot around the right stick; the mass will first rise, and then fall until the left stick hits the ground, etc. Eventually this rocking motion will cease, and the system will come to a halt. Let this occur at time T.

   For small θ, calculate, to leading order in $1/\theta$, the value of $T - t_0$.

   (You may use the small-angle formulas, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.)

   [Helpful hints in simplifying your answer:

   (a) $\ln(1-x) = -(x + x^2/2 + x^3/3 + x^4/4 + \cdots)$ for $-1 < x < 1$,

   (b) $1 + 1/3^2 + 1/5^2 + 1/7^2 + \cdots = \pi^2/8$.

   You get 7 points for getting a correct expression for $T - t_0$, and 3 points for simplifying it to show the explicit leading behavior in $1/\theta$; so you may not want to spend too much time on the simplification.]

6. A block with mass M (M is very large) initially sits on a frictionless plane, at a distance of 1 meter from the top of the plane [see the figure below]. The angle of inclination of the plane is θ. An elastic band connects the base of the block to the top of the plane. A cylindrical object, C, with mass m (m is very small), sits on the elastic band where it is connected to the plane. (The axis of C is parallel to the ground.)

   At $t = 0$, the block is allowed to slide down the plane, and C is allowed to roll down the elastic band.

   Assume: (1) the elastic band stretches uniformly, (2) M is large enough, and the elastic band is weak enough, so that the band has no effect on the motion of the block; the only purpose of the elastic band in this problem is to provide a surface for C to roll on (and the band, since it is stretching, may drag C along a bit), (3) there is sufficient friction between C and the elastic band so that C rolls on the band without slipping, and (4) m is small enough so that C has no effect on the stretching of the band.

   Let C have radius r and moment of inertia $I = \rho m r^2$.

   (a) (2 points) Show that if $\rho = 0$ (i.e., $I = 0$), then C will always remain 1 meter away from the block.

   (b) (8 points) Show that if $\rho > 0$ (i.e., $I > 0$), then

      i. the ratio of C's distance travelled along the plane to the block's distance travelled along the plane approaches 1, as $t \to \infty$;

      ii. as a function of t, the relative speed of the block and C behaves like $G(\rho)t^{-(1-\rho)/(1+\rho)}$, as $t \to \infty$, where $G(\rho)$ is a function of ρ which you are *not* required to determine.